Integer Linear models for efficient optical fiber sourcing on a wholesale market

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***Abstract*— In this paper, we study the problem of optimizing a telecommunication operator's sourcing strategy for optical fibers on a domestic wholesale market. We first propose a mixed integer non-linear formulation to formalize the problem, and prove the complexity of the problem in various frameworks. Then, based on the characterization of specific optimal solutions of a fiber demand repartition sub-problem, we propose two Integer Linear reformulations. Finally, we assess these formulations through extensive tests conducted on real-life data.**

I. Introduction

A. Context

The replacement of legacy copper fixed access networks with fiber-optical ones is undeniably one of the most challenging telecommunication infrastructure transformation of this century [7]. Such transformation appears for telecommunication operators as a major opportunity of relying on new fixed access networks that are more reliable (due to the properties of optical fibers) and more capable to sustain the growing demand from users experiencing ever-increasing bandwidth consuming services over time. Moreover, for governments, it represents an opportunity to provide every citizen with high-broadband access, with crucial concerns of numerical inclusiveness, which leads them to set ambitious deployment guidelines, see *e.g.* at the European and French level [2] [6].

For at least one decade, most European operators have engaged in massive roll-outs of their own optical fiber access networks as they consider such infrastructures as competitive assets. Nonetheless, such deployments incur tremendous investments costs, up to billions euros [5]. Since no commercial operator can afford such investments, they tend to favor areas with high financial profitability (typically urban areas which benefit from numerous customers and low distances between them) and leave aside non-profitable ones (typically rural areas). More precisely, in most European countries, regulators distinguish between areas where commercial operators are in charge of the deployment of optical access networks and areas where these deployments are left to Infrastructure Tier Operators (ITO). In these areas, along with ITO roll-outs, commercial operators' demand for optical access will grow and these will have to purchase from ITOs an increasing number of optical lines over time so as to satisfy their customers' demand: this is done by dint of a domestic wholesale market integrating renting and co-investment mechanisms specified by ITOs.

In this context, designing effective multi-year strategies for optical fibers sourcing so as to minimize costs while taking into account the specificity of their customers' demand evolution as well as their investments capabilities, is no easy task for telecommunication operators, but crucial given the strategic and economical stakes.

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B. State of the art

This strategic business problem, of prime importance for telcos, has been first introduced in [11], with a formal description of the decision problem leading to a mixed integer formulation refined by the design of valid inequalities.

While the literature related to optical access networks deployments has mainly focused on network design problems over the last decade [1] [10] [15], our problem belongs to the widely studied area of *multi-period planning*. Though embedding several specificities notably in terms of costs, its global structure is clearly related to the one of the *Knapsack Problem (KP)* [13] and more specifically to the *Generalized Assignment Problem (GAP)* [14]. Such problems have been widely studied with the design of exact integer programming based solution methods as well as approximation schemes [4] and dedicated heuristics [17].

C. Contributions

In this paper, we investigate the problem of *Optimizing Optical Fibers Sourcing* (OOFS) on a domestic wholesale market. Our main contributions are the following:

- we prove the strong NP-hardness (resp. weak NP-hardness) of the problem in the case of multiple co-investment committees (resp. one single committee).
- we characterize an optimal *demand repartition* strategy for fibers when co-investment decisions are known.
- we design two reformulations relying on this characterization: first one is inspired from the Generalized Assignment Problem (GAP) and the second one relies on the temporal sequences of co-investments.
- we prove the practical relevance of these formulations, assessing them on real-life data.

D. Outline of the paper

The remainder of this manuscript is organized as follows. Section II formally introduces our optimization problem, associated to an Integer Non-Linear formulation. Section III presents the main results on the problem complexity. In Section IV, we propose two reformulations for this problem that rely on the characterization of some optimal solutions of the fiber demand repartition sub-problem. We assess these models in Section V, before concluding in Section VI.

II. Problem description

We consider a domestic commercial operator (*e.g.* Orange France) that has to purchase fiber lines on the wholesale market over a discrete time-horizon $T = \{1, \dots, |T|\}$ so as to meet its future customers' demand for fiber access $d_{a,t},\ a \in A, t \in T$, with A a set of geographical areas, called zones, where the fiber access network deployment is left to Infrastructure Tier Operators (ITO). This planning horizon is mile-stoned by a set of (chronologically ordered) co-investment committees $C = \{1, \dots, |C|\}$, $t_c \in T$ denoting the time-period's index of the c^{th} committee. Thus, t_{c-1} stands for the time-period of the committee preceding committee $c \in C$ while $t_c - 1$ stands for the one preceding this committee. In addition, we note $t_0 = 0$ the time-period corresponding to the latest time before the planning horizon.

At each committee, the commercial operator can co-invest on a discrete set of amounts of fibers, called slices and indexed by I, corresponding to the acquisition of a percentage $Q_i,\ i \in I$ of the fiber lines deployed so far by the ITO. More precisely, at a given committee $c \in C$ taking place at time-period $t_c \in T$, purchasing a new (incremental) slice $i^{\delta}_{a,t_c} \in I$ for zone $a \in A$ leads to the acquisition a supplementary percentage $Q_{i^{\delta}_{a,t_c}}$ of the deployed fiber lines: the commercial operator then owns a cumulative co-investment slice $i^{tot}_{a,t_c} \in I$ corresponding to a percentage $Q_{i^{tot}_{a,t_c}}$ such that $Q_{i^{tot}_{a,t_c}} = Q_{i^{tot}_{a,t_0}} + \sum_{c' \in C | c' \leq c} Q_{i^{\delta}_{a,t_{c'}}}$, with i^{tot}_{a,t_0} denoting the initial cumulative co-investment slice. For convenience of notations, we will consider sets of slice percentages being multiples of a nominal percentage $Q \in]0, 100]$, *i.e.* $I = \{0, ..., \left\lfloor \frac{100}{Q} \right\rfloor\}$ with $Q_i = i \cdot Q\%, i \in I$. Such structure has the following advantages: (i) it allows to consistently describe both incremental and cumulative investment slices in a single set of reasonable size (ii) the null-investment is properly modeled by the slice indexed by 0 (iii) increments in investment percentages match with increments in slice indexes ($Q_i - Q_{i'}$ with $i' \leq i$ is equal to $Q_{i-i'}$). Incremental and cumulative investments are respectively modeled by binary variables $c^{\delta}_{a,t,i} \in \{0; 1\},\ \forall a \in A, t \in T, i \in I$ and $c^{tot}_{a,t,i} \in \{0; 1\},\ \forall a \in A, t \in T \cup \{0\}, i \in I$ equal to 1 if the respective incremental and cumulative slice is equal to i at time-period t for zone a, 0 otherwise. The *co-investment dynamics* can then be formulated as follows:

$$\sum_{i \in I} c^{tot}_{a,t,i} = 1,\ \forall a \in A, t \in T \cup \{0\}, \tag{1}$$

$$\sum_{i \in I} Q_i \cdot c^{tot}_{a,0,i} = Q_{i^{tot}_{a,t_0}},\ \forall a \in A, \tag{2}$$

$$\sum_{i \in I} c^{\delta}_{a,t,i} = 1,\ \forall a \in A, t \in T, \tag{3}$$

$$c^{\delta}_{a,t,0} = 1,\ \forall a \in A, t \in T \setminus (t_c)_{c \in C}, \tag{4}$$

$$\sum_{i \in I} Q_i \cdot c^{tot}_{a,t-1,i} + \sum_{i \in I} Q_i \cdot c^{\delta}_{a,t,i} = \sum_{i \in I} Q_i \cdot c^{tot}_{a,t,i},\ \forall a \in A, t \in T. \tag{5}$$

Constraints (1) impose the uniqueness of the cumulative investment slice at each time-period while Constraints (2) handle its initialization. Moreover Constraints (3-4) impose the uniqueness of the incremental investment slice (this one being constrained to be null outside committee time-periods). Finally, Constraints (5) ensure the consistency of the investments from one committee to the next.

Let $D_{a,t}$ be the number of fiber lines deployed by the ITO in zone $a \in A$ up to time-period $t \in T \cup \{0\}$: we suppose such time-series to be increasing w.r.t t for all zones. At any time-period $t \in T$, Capital Expenditures (CAPEX costs) are yield, first by co-invested fibers stemming from the new investment made at t, defined by $Q_{i^{\delta}_{a,t}} D_{a,t}$, and second, by the ones stemming from previous co-investments combined with new deployments made by the ITO between $t-1$ and t, defined as $Q_{i^{tot}_{a,t-1}}(D_{a,t} - D_{a,t-1})$: these amounts of lines are respectively charged with unitary CAPEX costs $CAP^{\delta}_{a,t}$ and $CAP^{tot}_{a,t}$. In addition, a CAPEX budget B_c is associated to any committee $c \in C$ which can be shared among zones $a \in A$ within a dedicated time window $P_c = [t_c, \dots, t_{c+1} - 1]$ (we define $t_{|C|+1} = T + 1$). Such *CAPEX budget* constraints can be formulated as follows:

$$\sum_{a \in A} \sum_{t \in P_c} \sum_{i \in I} \Big(Q_i (D_{a,t} - D_{a,t-1}) CAP^{tot}_{a,t} \cdot c^{tot}_{a,t-1,i} + Q_i D_{a,t} CAP^{\delta}_{a,t} \cdot c^{\delta}_{a,t,i} \Big) \leq B_c,\ \forall c \in C. \tag{6}$$

At each time-period $t \in T$ and for each zone $a \in A$, the commercial operator must then decide of the repartition of its customers' demand in fibers $d_{a,t}$ between co-invested and rented lines, knowing that it owns $\left\lfloor Q_{i^{tot}_{a,t}} D_{a,t} \right\rfloor$ co-invested lines on the zone. Introducing integer variables $u^{rent}_{a,t}$ and $u^{cofi}_{a,t}\ \forall a \in A, t \in T \cup \{0\}$ to model the number of users allocated to rented fiber lines versus co-financed ones, the *demand repartition* constraints can be formulated as follows:

$$u^{rent}_{a,t} + u^{cofi}_{a,t} = d_{a,t},\ \forall a \in A, t \in T \cup \{0\}, \tag{7}$$

$$u^{cofi}_{a,t} \leq D_{a,t} \sum_{i \in I} Q_i \cdot c^{tot}_{a,t,i},\ \forall a \in A, t \in T \cup \{0\}. \tag{8}$$

As for the planning cost to be minimized, for any zone $a \in A$ and time period $t \in T$, we distinguish between:

- a *renting cost* which is proportional to the number of rented fibers and can be formulated as $C^{rent}_{a,t} \cdot u^{rent}_{a,t}$.
- a *usage cost* (*e.g.* meant to cover for maintenance) proportional to the number of used co-invested fibers $u^{cofi}_{a,t}$. However, the unitary cost coefficient to be applied by the ITO depends on the cumulative investment slice $i^{tot}_{a,t}$. Noting this unitary cost $C^{cofi}_{a,t,i}$, the usage cost can be expressed as follows: $u^{cofi}_{a,t} \sum_{i \in I} C^{cofi}_{a,t,i} \cdot c^{tot}_{a,t,i}$.
- a *migration cost*, accounting for the ITO field intervention necessary to switch users from rented lines to co-invested ones and which has to be paid only at the first committee with a non-null investment. This one (if existing) is characterized by the unique time-period $t \in T$ where $c^{tot}_{a,t-1,0} - c^{tot}_{a,t,0}$ is equal to 1 (and not to 0). Moreover, if the customers' demand

for fiber decreases between time-periods $t-1$ and t, the amount of migrated lines is precisely equal to the increase in the number of used co-invested lines $u^{cofi}_{a,t} - u^{cofi}_{a,t-1}$, *i.e.* $u^{cofi}_{a,t}$ as this number is by definition equal to 0 at period $t-1$. Conversely, if the demand for fiber increases, the amount of migrated lines is equal to the decrease in rented lines $\max(0; u^{rent}_{a,t-1} - u^{rent}_{a,t})$. Denoting the unitary migration cost to be applied by $C^{mig}_{a,t}$ and the customers' demand evolution $d_{a,t} - d_{a,t-1}$, the migration cost can be expressed as $C^{mig}_{a,t}\big(\mathbb{1}_{\{d_{a,t}-d_{a,t-1}\le 0\}} u^{cofi}_{a,t} + \mathbb{1}_{\{d_{a,t}-d_{a,t-1}>0\}} \max(0; u^{rent}_{a,t-1} - u^{rent}_{a,t})\big)\big(c^{tot}_{a,t-1,0} - c^{tot}_{a,t,0}\big)$.

Problem statement. We aim to minimize the sum of the *renting*, *usage* and *migration* costs (over all time-periods and zones) while respecting the *co-investment dynamics* as well as the *demand repartition* and *CAPEX budget* constraints.

Objective function $obj = \sum_{a\in A}\sum_{t\in T}\big(C^{rent}_{a,t}\cdot u^{rent}_{a,t} + C^{mig}_{a,t}\cdot \big(\mathbb{1}_{\{d_{a,t}-d_{a,t-1}\le 0\}} u^{cofi}_{a,t} + \mathbb{1}_{\{d_{a,t}-d_{a,t-1}>0\}} \max(0; u^{rent}_{a,t-1} - u^{rent}_{a,t})\big)\big(c^{tot}_{a,t-1,0} - c^{tot}_{a,t,0}\big) + u^{cofi}_{a,t}\sum_{i\in I} C^{cofi}_{a,t,i}\cdot c^{tot}_{a,t,i}\big)$ and the sets of Constraints (1-8) define an Integer Non-Linear formulation for *OOFS*, nonlinearities stemming from products between bounded integer variables and binary ones and $\max$ operators. These can be linearized with classical techniques such as in [9]. The reader can refer to [11] for more details on the resulting Integer Linear formulation, referred to as (P^{base}).

III. Complexity study

Theorem 1. *OOFS Problem is*

- *weakly NP-hard for a single-committee setting.*
- *strongly NP-hard for a multiple-committee setting.*

Proof. *OOFS* is clearly in P as it admits a compact linear formulation (precisely (P^{base})).
Details of the proof are not given here due to the size of the article, but we can prove that the *Knapsack Problem* and the *3-Partition Problem* can respectively be reduced in polynomial time to a restricted version of the *OOFS*, these two reference problems being known respectively as weakly and strongly NP-hard (see [8]). □

IV. Mathematical framework

The main goal of this section is to provide two integer linear reformulations for *OOFS* (see. Sections IV-B and IV-C), that rely on the characterization of the costs of any optimal solution provided incremental and cumulative investment slices are known.

A. Optimal demand repartition characterization

We introduce $I^x = \{(i^\delta, i^{tot}) \in I^2 \mid i^\delta \le i^{tot}\}$ the set of *a priori* valid tuples of incremental and cumulative investment slices for any committee.

Assumption. In this section, we focus on a specific zone $a \in A$ and a specific committee $c \in C$ for which the incremental and cumulative investment slices for the committee time-period t_c are known and set to a specific tuple $(i^\delta, i^{tot}) \in I^x$.

Proposition 1. *The cumulated CAPEX cost over the committee window P_c yield by any feasible solution can be preprocessed as follows:*

$$CAP_{a,c,i^\delta,i^{tot}} = Q_{i^{tot}-i^\delta}(D_{a,t_c} - D_{a,t_c-1})CAP^{tot}_{a,t_c} + Q_{i^\delta} D_{a,t_c} CAP^{\delta}_{a,t_c} + \sum_{t\in P_c\setminus t_c} Q_{i^{tot}}(D_{a,t} - D_{a,t-1})CAP^{tot}_{a,t}.$$

Theorem 2. *A cost-optimal demand repartition strategy $\big(u^{cofi}_{a,t}, u^{rent}_{a,t}\big)$ can be computed for any time-period of the committee window $t \in P_c$ as follows:*

- $u^{cofi}_{a,t} =$
  <u>*case* $t = t_c, i^\delta > 0, i^{tot} = i^\delta$</u>
  $$\begin{cases} 0 \text{ if } C^{rent}_{a,t} \le C^{cofi}_{a,t,i^{tot}}, \\ \min(d_{a,t}; Q_{i^{tot}} D_{a,t}) \text{ if } C^{cofi}_{a,t,i^{tot}} + C^{mig}_{a,t} \le C^{rent}_{a,t}, \\ \min(\max(d_{a,t} - d_{a,t-1}; 0); Q_{i^{tot}} D_{a,t}) \text{ otherwise.} \end{cases}$$
  <u>*otherwise:*</u>
  $$\begin{cases} 0 \text{ if } C^{rent}_{a,t} \le C^{cofi}_{a,t,i^{tot}}, \\ \min(d_{a,t}; Q_{i^{tot}} D_{a,t}) \text{ otherwise.} \end{cases}$$
- $u^{rent}_{a,t} = d_{a,t} - u^{cofi}_{a,t}, \; t \in T, a \in A.$

Proof. Noticing that the first committee with a non-null investment is characterized by the condition $i^\delta > 0$ and $i^{tot} = i^\delta$, the computation of an optimal *demand repartition* strategy of $d_{a,t}$ between $u^{cofi}_{a,t}$ and $u^{rent}_{a,t}$ for any zone $a \in A$ and any given time-period $t \in T$ taken separately is straight-forward, driven by a comparison of unitary costs. The fact that such strategy is globally optimal (considering the whole time-horizon T) stems from two complementary considerations: first, users allocated to co-financed fibers can be switched to rented ones at any time with null costs; second, except at the first committee with a non-null investment, rented lines can be migrated to co-financed ones at zero-cost at any time-period. These conditions are sufficient to erase any time-dependency between demand repartition decisions. □

Corollary 1. *If time-period t_c corresponds to the first committee with a non-null investment, then the number of migrated fibers is equal to* $\mathbb{1}_{C^{cofi}_{a,t_c,i^{tot}} + C^{mig}_{a,t_c} \le C^{rent}_{a,t_c}} \cdot \max\big(0; \min(d_{a,t_c}; Q_{i^{tot}} D_{a,t_c}) - \max(0; d_{a,t} - d_{a,t-1})\big)$.

Proof. Migration occurs only if the sum of the unitary usage and migration costs $C^{cofi}_{a,t_c,i^{tot}} + C^{mig}_{a,t_c}$ is lower to the unitary renting cost C^{rent}_{a,t_c}. In this case, the number of migrated fibers at t_c can be computed as the number of co-financed and used lines at t_c, *i.e.* $\min(d_{a,t_c}; Q_{i^{tot}} D_{a,t_c})$ (see Theorem 2), minus the increment in fiber demand between $t_c - 1$ and t_c as these fibers are directly allocated to co-financed lines. □

Proposition 2. *The cumulated solution cost over the committee window P_c yield by any feasible solution having an* *optimal demand repartition strategy within P_c with respect to the tuple of co-investment slices (i^δ, i^{tot}) can be preprocessed as follows:*

$$
\begin{aligned}
C_{a,c,i^\delta,i^{tot}} = \sum_{t\in P_c} \Big(& \mathbb{1}_{\{C^{rent}_{a,t} \leq C^{cofi}_{a,t,i^{tot}}\}} C^{rent}_{a,t} d_{a,t} \\
& + \mathbb{1}_{\{C^{rent}_{a,t} > C^{cofi}_{a,t,i^{tot}}\}} C^{cofi}_{a,t,i^{tot}} \min(d_{a,t}; Q_{i^{tot}} D_{a,t}) \\
& + \mathbb{1}_{\{C^{rent}_{a,t} > C^{cofi}_{a,t,i^{tot}}\}} C^{rent}_{a,t} \max(0; d_{a,t} - Q_{i^{tot}} D_{a,t}) \Big) \\
+ & \mathbb{1}_{\{C^{cofi}_{a,t_c,i^{tot}} \leq C^{rent}_{a,t_c} \leq C^{cofi}_{a,t_c,i^{tot}} + C^{mig}_{a,t_c} \text{ and } i^\delta > 0 \text{ and } i^{tot} = i^\delta\}} \\
& \cdot \big(\min(Q_{i^{tot}} D_{a,t_c}; \max(d_{a,t} - d_{a,t-1}; 0)) \\
& - \min(d_{a,t_c}; Q_{i^{tot}} D_{a,t_c}) \big) \cdot \big(C^{cofi}_{a,t_c,i^{tot}} - C^{rent}_{a,t_c} \big) \\
+ & \mathbb{1}_{\{C^{rent}_{a,t_c} > C^{cofi}_{a,t_c,i^{tot}} + C^{mig}_{a,t_c} \text{ and } i^\delta > 0 \text{ and } i^{tot} = i^\delta\}} \cdot C^{mig}_{a,t_c} \\
& \cdot \max\big(0; \min(d_{a,t_c}; Q_{i^{tot}} D_{a,t_c}) - \max(0; d_{a,t} - d_{a,t-1})\big).
\end{aligned}
$$

Proof. Such cost is the one of the optimal demand repartition strategy provided in Theorem 2. □

B. GAP-oriented reformulation

We introduce binary variables $x_{a,c,i^\delta,i^{tot}}$, $a \in A$, $c \in C \cup 0$, $(i^\delta, i^{tot}) \in I^x$ equal to 1 if an incremental investment of $Q_{i^\delta}\%$ is made at time-period t_c, leading to a cumulative percentage of $Q_{i^{tot}}\%$, 0 otherwise (we initizalize $x_{a,0,i^\delta,i^{tot}}$ to 1 for $(i^\delta, i^{tot}) = (0, i^{tot}_{a,t_0})$, 0 otherwise). We then propose a GAP-oriented formulation, denoted by *(P^x)*:

$$
\begin{aligned}
\min_x \quad & \sum_{a\in A} \sum_{c\in C} \sum_{(i^\delta,i^{tot})\in I^x} C_{a,c,i^\delta,i^{tot}} \cdot x_{a,c,i^\delta,i^{tot}} \\
& \sum_{a\in A} \sum_{(i^\delta,i^{tot})\in I^x} CAP_{a,c,i^\delta,i^{tot}} \cdot x_{a,c,i^\delta,i^{tot}} \leq B_c, \\
& \hfill \forall c \in C, \quad (1x) \\
& \sum_{(i^\delta,i^{tot})\in I^x} x_{a,c,i^\delta,i^{tot}} = 1, \qquad \forall a \in A, c \in C, \quad (2x) \\
& \sum_{(i^\delta,i^{tot})\in I^x} (Q_{i^{tot}} - Q_{i^\delta}) \cdot x_{a,c,i^\delta,i^{tot}} = \\
& \sum_{(i^\delta,i^{tot})\in I^x} Q_{i^{tot}} \cdot x_{a,c-1,i^\delta,i^{tot}}, \forall a \in A, c \in C, \quad (3x) \\
& x_{a,c,i^\delta,i^{tot}} \in \{0;1\}, \ \forall a \in A, c \in C, (i^\delta, i^{tot}) \in I^x.
\end{aligned}
$$

The objective is to minimize the cumulative operational costs for all zones over the time-horizon. Constraints (1x) ensures the respect of the budget to be shared among all zones at each committee. Constraints (2x) impose the uniqueness of both incremental and investment percentages at each committee (time-period), while Constraints (3x) ensure the consistency of investments from one committee to the next.

C. Sequence-oriented reformulation

let $I^y_a = \Big\{ (i^{tot}_0, \ldots, i^{tot}_{|C|}) \in I^{|C|+1} \ \Big| \ i^{tot}_0 = i^{tot}_{a,t_0} \text{ and } i^{tot}_{c-1} \leq i^{tot}_c, \ \forall c \in C \Big\}$ be the set of *a priori* valid co-investment plans for any zone $a \in A$, represented by sequences of increasing cumulative investment slices for successive committees.

We introduce binary variables $y_{a,s}$, $\forall a \in A, s \in I^y$ equal to 1 if the cumulative investment sequence for zone a is equal to s (0 otherwise), on which we base the following sequence-based formulation for *OOFS*, denoted by (P^y):

$$
\begin{aligned}
\min_y \quad & \sum_{a\in A} \sum_{\substack{s\in I^y_a \\ s=(i^{tot}_0,\ldots,i^{tot}_{|C|})}} y_{a,s} \sum_{c\in C} C_{a,c,(i^{tot}_c - i^{tot}_{c-1}),i^{tot}_c} \\
& \sum_{a\in A} \sum_{\substack{s\in I^y_a \\ s=(i^{tot}_0,\ldots,i^{tot}_{|C|})}} CAP_{a,c,(i^{tot}_c - i^{tot}_{c-1}),i^{tot}_c} \cdot y_{a,s} \leq B_c, \\
& \hfill \forall c \in C, \quad (1y) \\
& \sum_{s\in I^y_a} y_{a,s} = 1, \qquad \forall a \in A, \quad (2y) \\
& y_{a,s} \in \{0;1\}, \ \forall a \in A, s \in I^y_a.
\end{aligned}
$$

The objective is to minimize the cumulative operational costs for all zones over the whole time-horizon: key point in the expression is the re-use of operational costs by committee window parameters as defined in Proposition 2, noticing that the incremental investment slice of any committee c is the difference between the cumulative investment slices of committees c and $c-1$ due to the format of the investment percentages mentioned in Section II. Same observation holding for CAPEX costs, Constraints (1y) ensure the respect of the budget to be shared among all zones at each committee. Lastly, Constraints (2y) ensure the choice of a single valid multi-period investment plan (sequence) for each zone.

Remark 1. *The number of non-decreasing sequences of $|C|$ elements taken from a set of $|I|$ distinct ones and allowing repetitions is given by (see. Theorem 2.21 in [3]):*

$$
\binom{|C| + |I| - 1}{|C|} = \frac{(|C| + |I| - 1)!}{(|I| - 1)!|C|!}
$$

As a consequence, (P^y) is not a compact formulation.

D. Polyhedral analysis

We note respectively P^{base}, P^x and P^y the polyhedra associated to formulations (P^{base}), (P^x) and (P^y).

Proposition 3. *The optimal value of the linear relaxation associated to (P^x) formulation is lower than or equal to the one associated to (P^y).*

Proof. Details of the proof are not given here due to the size of the article, but this result relies on the fact that, from any solution $y \in P^y$, we can derive a solution $x \in P^x$ of same cost as y, such as for all $a \in A, c \in C, (i^\delta, i^{tot}) \in I^x$: $x_{a,c,i^\delta,i^{tot}} = \sum_{\substack{(i^{tot}_0,\ldots,i^{tot}_{|C|})\in I^y_a \\ i^{tot}_c = i^{tot} \ \& \ i^{tot}_{c-1} = i^{tot} - i^\delta}} y_{a,(i^{tot}_0,\ldots,i^{tot}_{|C|})}$. □

Proposition 4. *The optimal value of the linear relaxation associated to (P^{base}) formulation is lower than or equal to the one associated to (P^x).*

Proof. Details of the proof are not given here due to the size of the article, but this result relies on the fact that, from any solution $x \in P^x$, we can derive a solution $cu = (c^\delta, c^{tot}, u^{rent}, u^{cofi}) \in P^{base}$ of same cost as x, as follows:

$$\begin{cases} c^\delta_{a,t_c,i} = \sum_{(i^\delta,i^{tot})\in I^x | i^\delta = i} x_{a,c,i^\delta,i^{tot}} \ \forall a \in A, c \in C, i \in I, \\ c^\delta_{a,t,0} = 1 \text{ and } c^\delta_{a,t,i\neq 0} = 1 \ \forall a \in A, t \in T \setminus (t_c)_{c\in C}, \end{cases}$$

$$\begin{cases} c^{tot}_{a,t_c,i} = \sum_{(i^\delta,i^{tot})\in I^x | i^{tot} = i} x_{a,c,i^\delta,i^{tot}} \ \forall a \in A, c \in C, i \in I, \\ c^{tot}_{a,t,i} = c^{tot}_{a,t_c,i} \ \forall a \in A, t \in T \setminus (t_c)_{c\in C}, i \in I. \end{cases}$$

Variables (u^{rent}, u^{cofi}) are then given by Theorem 2. $\square$

V. EXPERIMENTATION

The three formulations were implemented in Python language, relying on the Pyomo library [12] and default branch-and-cut from CPLEX 12.9 [16] for ILP solving.

A. Tests setting

The experiments were conducted on a server equipped with 256 GB of RAM and 32 CPU parallel threads. The maximum CPU time for each run was set to 1 hour (3600 seconds), while ILP optimality gap was set to 10^{-5}.
To assess our models, we relied on a testbed of real-life instances from the French context, with a number of zones $|A|$ in range $\{25, 50, 250, 750\}$, and with time-horizons composed of $|T| \in \{12, 36, 72, 108, 180\}$ months being milestoned with one investment committee per year (*i.e.* $|C| \in \{1, 3, 6, 9, 15\}$). Furthermore, we based the investment structure on $Q = 5$, *i.e.* $I = \{0, ..., 20\}$ with $Q_i = i{\cdot}5\%, i \in I$ and assumed null initial cumulative co-investments. Endly, we consider realistic CAPEX budgets built as percentages $\alpha \in \{25\%, 50\%, 75\%\}$ of the minimum CAPEX cost necessary to avoid all fiber renting. More precisely, a cumulative CAPEX cost (over the whole time-horizon) is obtained by solving any model without CAPEX constraints: this quantity is multiplied by the parameter α and then uniformly distributed across the $|C|$ committees.

B. Limitation of the (P^{base}) formulation

This first computational study is restricted to small instances with $|C| \in \{1, 3\}$ and $|A| \leq 50$, and computational results related to (P^{base}) are stored in Table I.

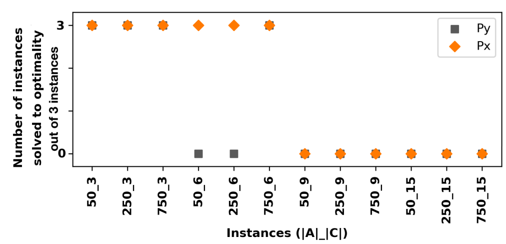
Instances			(P^{base})		
$\vert A\vert$	$\vert C\vert$	α	**NbrNd**	**Gap**	**TT**
25	1	25%	117612	0,00%	906
	1	50%	980008	0,02%	3600
	1	75%	607088	0,00%	2448
50	1	25%	59614	2,66%	3600
	1	50%	59368	3,01%	3600
	1	75%	56475	2,30%	3600
25	3	25%	30976	5,63%	3600
	3	50%	12249	14,10%	3600
	3	75%	8014	22,65%	3600
50	3	25%	8408	7,33%	3600
	3	50%	10997	19,06%	3600
	3	75%	1648	33,57%	3600

TABLE I: Numerical results for (P^{base}) formulation.

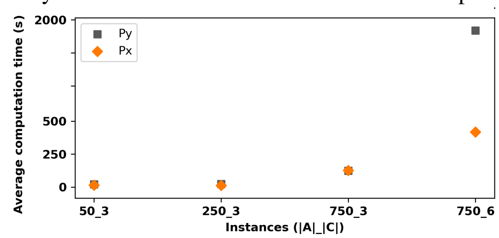
Main observation is that only 2 out of 12 instances are solved to optimality when using this formulation, with an average computation time (`TT`) of 3280 seconds. Moreover for the 10 remaining instances, we observe an average final gap (`Gap`) of 9.19% which clearly tends to increase with the number of zones (reaching up to 33.57%). In contrast, figures related to (P^x) and (P^y) formulations are not detailed here, but all these instances are solved to optimality within a few seconds when using these formulations, showing their advantages over (P^{base}) for handling real-life instances.

C. Comparative analysis of (P^x) and (P^y)

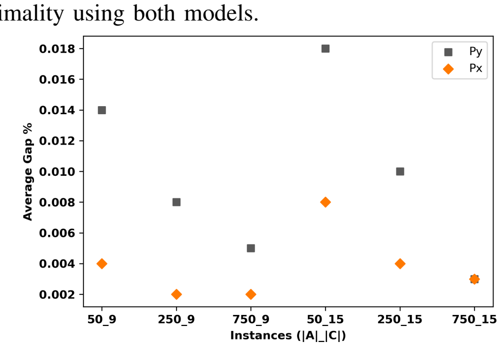
This study is conducted on instances with $|A| \in \{50, 250, 750\}$ and $|C| \in \{3, 6, 9, 15\}$. Figure 1a illustrates the capability of the models to solve instances to optimality within time limit. Results show that using (P^x) enables us



(a) Analysis of the number of instances solved to optimality.



(b) Analysis of the computation time on instances solved to optimality using both models.



(c) Analysis of the (relative) final gap on instances that are not solved to optimality by both models.

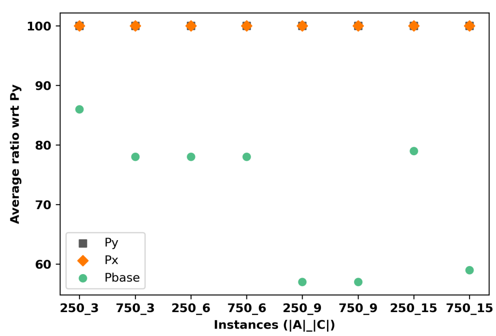
Fig. 1: Comparative analysis of (P^x) and (P^y).

to solve more instances to optimality than using (P^y), with 6 instances solved with (P^x) while not with (P^y). Additionally, as shown in Figure 1b which focuses on instances solved to optimality with both formulations, we observe that the average computation time is significantly lower when using (P^x) compared to (P^y) (143 vs. 521 seconds). For instances with more than 9 committees, convergence is observed for none of the formulations, highlighting an increased computational complexity as the number of committees grows.

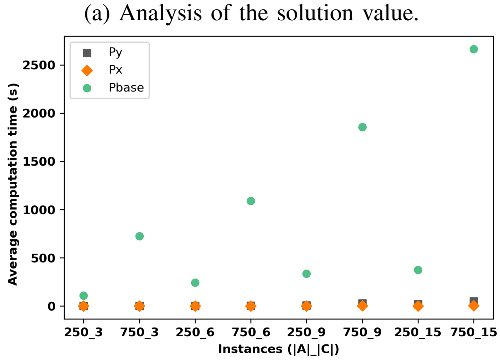
However, both formulations were able to provide highly efficient solutions, with average gap percentages of 0.0038% for (P^x) and 0.0097% for (P^y), as shown in Figure 1c. As a conclusion, (Px) proves to be the most efficient formulation to handle real-life instances, even if (Py) almost competes, except for instances with 15 committees: for these instances, the building phase of the model exceeded the time limit, which is in line with Remark 1.

D. *Continuous relaxation analysis*

In this section, we analyze the quality of the linear relaxation of our 3 formulations on large instances, *i.e.* with $|A| \in \{250, 750\}$ and $|C| \in \{3, 6, 9, 15\}$. First, we present in Figure 2a the ratio between the relaxation solution value of any formulation and the one of (P^y), which is proven to be the higher one in Section IV-D. We observe that the linear relaxation value of (P^{base}) is significantly lower than (P^x) and (P^y) (average ratio of 71.5%) while (P^x) and (P^y) achieved the same optimal relaxation value on all instances. Moreover, Figure 2b shows that (P^x) significantly outperforms (P^y) and (P^{base}) in terms of computation time, with an average computation time of 1.35 seconds versus 46.91 and 2663.92 seconds, respectively.



(a) Analysis of the solution value.



(b) Analysis of the computation time.

Fig. 2: Quality of the linear relaxation of the 3 models.

VI. Conclusions and perspectives

In this paper, we study the decision problem, for a telecommunication operator, of purchasing optical fibers on a wholesale domestic market, in order to supply its future customers' demand in geographical areas where it does not deploy its own optical access network. Designing efficient sourcing strategies, by leveraging renting and co-investing at light of incurred investment and operational costs, is of prime economical importance for domestic operators.

In this work we perform an in-depth analysis of the problem complexity within different settings and propose several Integer Linear (re)formulations for the problem, that are assessed through extensive numerical tests performed on real-life data from the French context.

These tests highlight the practical relevance of the proposed GAP-oriented and sequence-based reformulations, with limitations occurring for the latter when the number of committees significantly increases: this suggests that designing column generation based solving approaches for this formulation should be a valuable research axis. Moreover this work deals with a deterministic setting which could be challenged as future customers demand, despite based on accurate forecasts from marketers, cannot be known with certainty: considering an uncertain setting thus appears as a second natural research avenue for this work.

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