Optimal Inventory Pressure Control for the High-Frequency Market Making

Alexandr Rzhevkin[1], Vadim Azhmyakov[2], *Senior Member, IEEE*, and Luz Adriana Guzman Trujillo[3]

Abstract— **This paper introduces a novel Inventory Pressure Control (IPC) framework for High-Frequency Trading (HFT). Moving beyond traditional stochastic optimal control models, our approach utilizes a specific parametric nonlinear feedback control to define the necessary bid/ask quotes. It incorporates some market indicators such as volatility, order flow imbalance, and toxic order flow. We establish the local stochastic stability of the resulting closed-loop inventory dynamics, proving that the process remains bounded in probability. This local stability framework provides a robust foundation for an integrated profit-risk optimization using the Dynamic Programming (DP) principle. We demonstrate the local profitability of the proposed optimal IPC strategy and validate our theoretical results through an illustrative example.**

I. Introduction

The rapid evolution of the modern financial engineering and the dominance of HFT have necessitated more robust and adaptive market-making frameworks. Conventional approaches, most notably the influential stochastic control models pioneered by Avellaneda and Stoikov in [2], often rely on specific assumptions regarding price processes and execution intensities. While mathematically elegant, these models can struggle with the extreme latency requirements and highly non-stationary nature of modern financial markets.

Let us recall that the Avellaneda-Stoikov approach is fundamentally rooted in stochastic optimal control theory. This classical framework typically aims to solve the market maker utility maximization problem. In our paper, we propose an alternative to these standard stochastic-control trading models. Our methodology applies control theory to the domain of technical analysis, extending the general feedback control framework for financial markets discussed in [3], [4], [6]–[9], [11], [13]. It is also important to note that there are currently various control-theoretical applications to the modern financial engineering (see, e.g., [4], [8], [10], [13] and the references therein).

The market-making strategy proposed in this paper is based on a novel Inventory Pressure Control (IPC) framework. Rather than relying on complex price-drift estimations, our approach leverages control-theoretic technical analysis with an active feedback type control of the market maker position. The IPC constitutes a parametric feedback control for inventory dynamics and incorporates some key market indicators, such as volatility, order flow imbalance, queue risk, and the so-called toxicity proxy (see e.g., [12], [15]). Furthermore, the inventory dynamics are assumed to be governed by some general stochastic compound processes. The proposed IPC framework directly determines the necessary bid/ask quotes, thereby exerting a direct influence on the resulting Profit and Loss (PnL) profile of the trading strategy. We next study the local stochastic boundedness and stochastic stability properties of the resulting closed-loop inventory dynamics. We establish local stability in the mean and, consequently, local stability in probability for the aforementioned dynamics. In fact, the resulting (locally) stabilizing IPC strategy guarantees that the inventory dynamic process remains within a bounded neighborhood of a target with high probability. This local asymptotic stability property of the closed-loop dynamics evidently implies a significant damping of inventory risk within the HFT framework and, furthermore, provides a robust basis for hedging strategies.

Notably, the stability conditions derived in our paper do not dictate a "rigid" trading execution policy. Rather, they define a flexible local stability boundaries within which PnL-driven optimization can be effectively pursued. We next leverage the obtained stable IPC framework to develop an integrated optimization approach that simultaneously addresses PnL maximization and risk awareness. We apply the celebrated Dynamic Programming (DP) approach for the corresponding optimization procedure (see, e.g., [1], [5]). In this paper, we derive the necessary Bellman equation and discuss the mathematical foundations of this optimization approach. A concrete large-scale application of the proposed procedure is deferred to future work. The resulting locally stable and optimal IPC strategy is shown to possess a local profitability property.

The remainder of this paper is organized as follows: Section 2 provides a formal problem formulation and describes the detailed structure of the IPC market-making strategy. In Section 3, we study the boundedness and local stability properties of the closed-loop inventory dynamics. We define the necessary bid/ask quotes and derive explicit conditions for local asymptotic stability in mean and in probability. Section 4 presents the PnL-risk based optimization approach for the locally stable IPC strategy. We derive the discrete-time Bellman equation and establish the local profitability property of the resulting IPC strategy. This section also includes an illustrative numerical example. Section 5 summarizes our paper.

[1]Alexandr Rzhevkin is with AIP Algorithmic Trading Company, 3095 Limassol, Republic of Cyprus `alex@aip.tech`

[2]Vadim Azhmyakov is with the Department of Mathematics, SRM University-AP, Amaravati 522240, Andhra Pradesh, India `vadim.a@srmap.edu.in`

[3]Luz Adriana Guzman Trujillo is with the LARIS, University of Angers, 49035 Angers, France `luz.guzmantrujillo@etud.univ-angers.fr`

II. Inventory Pressure Based Control in High-Frequency Trading

Let us consider a high-frequency market maker operating on a stock and assume that the size of his orders is small enough to consider price dynamics exogenous. We deal with a discrete-time dynamics of the financial markets under consideration and use the following notation of the stock ticks $t \in \mathbb{N}$. Due to the stochastic nature of the stock price p, we next introduce the probability space $\{\Omega, \mathcal{F}, Prob\}$ equipped with the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$. A stock price $p(\omega, t)$ can formally be considered as a measurable function $p : \Omega \times \mathbb{Z}_+ \to \mathbb{R}$. Recall that $p(\cdot, \cdot)$ is usually modelled by a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ of the probability space Ω. A concrete realization of the market prices $p_t = p(\cdot, t)$ of an asset at $(T+1)$ time points constitutes a series $\{p_t\}_{t=0,1,...,T}$.

The inventory $q_t \in \mathbb{Z}$ is the (signed) quantity of shares a market maker holds. This value varies with the execution of the limit orders placed by the market-maker. In parallel to q_t, we next consider some important market parameters, namely, the Order Flow Imbalance OFI_t, the Queue Risk QR_t and the so-called Toxicity Proxy TX_t. We refer to [4], [14] for the market related explanations of these parameters and also for the corresponding calculation rules. The market parameter vector under consideration is defined as:

$$\theta_t := \left(OFI_t, QR_t, TX_t, \sigma_t^2\right)^T,$$

the short-term volatility σ_t^2 is calculated via the conventional EWMA (Exponentially Weighted Moving Average) estimator (see e.g., [19]):

$$\sigma_t^2 := (1-\lambda)r_t^2 + \lambda\sigma_{t-1}^2, \ \lambda \in (0.7, 0.9).$$

Here, $r_t := \ln(p_t/p_{t-1}), \ t = 1, ..., T$ is the log-return. The specific choice of the decay factor λ ensures that the estimator remains highly responsive to transient market shocks, which is a critical requirement for the HFT framework. Note that the components of the vector θ_t are independent of the agent's inventory q_t and describe exclusively the aggregate liquidity dynamics and the market's information flow. Note that in the real market situation the parameters QR_t and TX_t may depends on the agent's action. Our paper adopts the full exogeneity assumption. This condition serves as an adequate first-order approximation for the market dynamics involved.

The market maker will continuously define the bid and ask quotes defined respectively as b_t and a_t and will buy and sell shares according to the rate of arrival of market orders at the quoted prices. We now generalize the approach proposed by Guéant [14] and introduce the following discrete-time inventory dynamics:

$$q_{t+1} = q_t + F_{t+1}^b(b_t, x_t) - F_{t+1}^a(a_t, x_t), \qquad (1)$$

where $x_t := (q_t, \theta^T)^T$ and $F_t^b, \ F_t^a$ are the stochastic point processes giving the number of shares the market maker respectively bought and sold. We next assume that F_t^b and F_t^a are the independent stochastic compound processes

$$F_{t+1}^b(b_t, x_t) = Z_{t+1}^b(b_t, x_t)Y_{t+1}^b,$$
$$F_{t+1}^a(a_t, x_t) = Z_{t+1}^a(a_t, x_t)Y_{t+1}^a,$$

where $Z_t^b, \quad Z_t^a \in \{0, 1\}$ are Bernoulli processes and $Y_t^b, \ Y_t^a$ are some admissible (stochastic) marks. We further assume that the processes $Z_t^b, \ Y_t^b, \ Z_t^a$ and Y_t^a are mutually independent. We also assume that stochastic processes Y_t^b and Y_t^a have uniformly bounded first and second moments

$$E[(Y_t^b)] = E[(Y_t^a)] = \mu_t < M < \infty,$$
$$E[(Y_t^b)^2] < M_b < \infty, \ E[(Y_t^a)^2] < M_a < \infty, \ t \in \mathbb{N},$$

where $E[\cdot]$ denotes the operator of mathematical expectation. Let us note that for a concrete "worst-case" modelling and simulation one can consider some truncated Cauchy processes $Y_t^b, \quad Y_t^a$ in equation (1). While the standard Cauchy distribution is known for having undefined moments, one has to deal with the truncated Cauchy processes and restrict the above processes to a compact interval. By capping the support of the distribution at a sufficiently large threshold, we ensure that the processes possess uniformly bounded first and second moments (see e.g., [19]). This allows the model to capture the characteristic heavy tails of market shocks while remaining mathematically tractable for the technical analysis. By employing a truncated support, we bridge the gap between empirical market reality and mathematically rigorous requirements. We also assume that the aforementioned distributions are shifted, implying non-zero mean $\mu_t \neq 0$. This assumption is consistent with a non-zero Order Flow Imbalance $OFI_t \neq 0$.

The initial inventory dynamics (1) can now be rewritten:

$$q_{t+1} = q_t + Z_{t+1}^b(b_t, x_t)Y_{t+1}^b - Z_{t+1}^a(a_t, x_t)Y_{t+1}^a. \qquad (2)$$

To align the inventory dynamic model (2) with empirical market conditions, we impose a box constraint on variable q_t. Specifically, we introduce a feasible set $[q_{min}, q_{max}]$ for inventory $q_{min} \leq q_t \leq q_{max}$, where q_{min} and q_{max} represent the lower and upper operational bounds, respectively. Similar to [2], we now define the Bernoulli probabilities $P_t^b, \quad P_t^a$ associated with $Z_t^b, \quad Z_t^a$ using the exponential intensities

$$P_t^b = A_t e^{-k_b b_t}, \ P_t^a = A_t e^{-k_a a_t} \qquad (3)$$

Here A_t and $k_b, \ k_a$ are positive constants that characterize the liquidity of the stock under consideration. Note that for some positive quotes b_t and a_t that implies the following: the closer to the reference price an order is posted, the faster it will be executed.

As mentioned in Introduction, we follow here a concept of the IPC based feedback-type market strategy (see e.g., [3], [4], [6]). Concretely, we consider the following saturation-involved control design

$$u_t = u_{max}(t)\tanh(Pr_t(x_t, \eta_t)/s_t),$$
$$u_{max}(t) \in U \subset \mathbb{R} \ \forall t \in \mathbb{N}, \qquad (4)$$

where Pr_t is an Inventory Pressure (IP) and s_t is a sensitivity. Let the admissible region $U \subset \mathbb{R}$ for control u_{max} be compact. The IP Pr_t in (4) can be interpreted as a specific linear combination of the inventory q_t and market parameters from θ_t:

$$Pr_t(x_t, \eta_t) := \alpha_t q_t + \beta_t OFI_t + \gamma_t QR_t + \delta_t TX_t, \qquad (5)$$

where $\eta_t := (\alpha_t, \beta_t, \gamma_t, \delta_t)^T$ is a vector of coefficients (weights) with $\alpha \in [\alpha_{min}, \alpha_{max}] \subset \mathbb{R}$. We next assume that $\beta_t, \gamma_t, \delta_t$ in η_t are convex coefficients, namely,

$$\beta_t \geq 0, \gamma_t \geq 0, \delta_t \geq 0, \ \beta_t + \gamma_t + \delta_t = 1$$

for all $t \in \mathbb{N}$. To combine for the potentially incompatible scales of the market variables in (5), we also assume the elements of η_t (coefficients in (5)) are scale-adjusting weights. In other words, the weight vector η_t constitutes a re-scaling operator, where each coefficient compensates for the underlying dimensionality of the respective market metric to ensure a balanced contribution to Pr_t. Note that sensitivity parameter s_t in (4) determines how "sensitive" the control u_t in (4) is to changes in the inventory pressure Pr_t and shows how quickly it saturates. Note that for a small time interval we have $s_t = l\sigma_t$ with a coefficient $l \in \mathbb{R}_+$.

It is clear that a positive inventory $q_t > 0$ generates a corresponding positive component of the inventory pressure and implies a long exposure risk. Similarly, a positive OFI_t in (5) indicates prevailing buying pressure in the market. Within the proposed control framework (4)-(5), an increase in the IP $Pr_t(x_t, \eta_t)$ leads to a corresponding adjustment of the bid and ask levels. We will discuss the specific application of this control scheme to the design of the bid/ask quotes in Section III. The resulting alignment ensures that the market maker stays ahead of the trend rather than providing liquidity at obsolete prices.

On the other hand, coefficient γ_t scales the impact of a market maker position in the order book. It must be calibrated so that a deteriorating queue position (higher risk) appropriately widens the spread or adjusts the offset to maintain the target fill probability. Finally, $\delta_t TX_t$ in (5) expresses a weighted toxicity proxy (e.g., VPIN). A high δ_t ensures that when toxic flow is detected, the IP increases sharply and pushes the trading strategy into a defensive mode.

III. Local Stability of the Inventory Dynamics

A higher IP Pr_t in (5) implies a more aggressive control signal u_t in (4). This fact motivates a specific quote construction that shifts the market maker orders (e.g., tightening the spread or adjusting the offset) to improve the execution priority and mitigate the risk. Following this concept, we are now ready to determine concretely the bid / ask quotes:

$$\begin{aligned} b_t &= mid_t - Sp_t/2 + \nu_t^b u_t, \\ a_t &= mid_t + Sp_t/2 + \nu_t^a u_t. \end{aligned} \tag{6}$$

where $\nu_t^b, \ \nu_t^a \in R_+$. Here mid_t is a mid price associated with the pairs of best bid and best ask prices at each time step $t = 0, ..., T$ and Sp_t is so-called effective spread

$$Sp_t(x_t, \eta_t) := Sp_0 + \rho |Pr_t(x_t, \eta_t)|. \tag{7}$$

Here Sp_0 is a base spread (for example, the bid-ask spread), and the component $\rho|Pr_t(x_t, \eta_t)|, \ \rho \in \mathbb{R}$ expands the basic spread as the IP Pr_t (driven by inventory, toxicity, or order flow imbalance) increases.

When the absolute value $|Pr_t|$ is near zero, the effective spread Sp_t converges toward Sp_0. This implies symmetric quoting and tight spreads. The dynamic inventory system reaches its equilibrium when the corresponding inventory pressure Pr_t is approximately zero. This occurs when the state (inventory) q_t and the components of vector θ_t, namely, the market parameters in (5) are all at their target levels. For the inventory q_t that means that the position is at or near the target (usually zero). The equilibrium condition for OFI_t can be described as follows: the buying and selling pressure from the market are balanced. Correspondingly, for the toxicity TX_t the target mentioned above means that no significant signal of "toxic" or informed flow is given. The above equilibrium condition motivates the natural stability requirement for the proposed feedback-type control (4) associated with the bid / ask quotes (6). Roughly speaking, the objective of the proposed control law is not only to maximize profit, but also to guarantee a stable inventory dynamics that constantly pull the system state q_t back toward this benchmark. As soon as the system drifts away from this benchmark, the controller immediately shifts the quotes and pulls the system back toward equilibrium.

Taking into consideration the above quotes rule (6), the inventory dynamics (2) - (3) closed by the parametric feedback control (4) - (5) constitutes a stochastic closed-loop system. We next call it a Feedback Inventory Dynamics (FID). Moreover, taking into account the recursive structure of the inventory dynamics (2) and the pressure-based feedback control framework (4) - (5), we conclude that the resulting FID generates a Markov chain $\{q_t\}_{t\in\mathbb{N}}$. To analyze the asymptotic boundedness and the asymptotic stability of the given FID, we now consider the mean dynamics. Applying the conditional expectation $E[\cdot \mid q_t]$ to the state (inventory) equation (2), we obtain

$$\begin{aligned} &E[q_{t+1} \mid q_t] = q_t + \\ &E[Z^b_{t+1}(b_t, x_t)]E[Y^b_{t+1}] - E[Z^a_{t+1}(a_t, x_t)]E[Y^a_{t+1}] = \\ &q_t + \mu_t(E[Z^b_{t+1}(b_t, x_t)] - E[Z^a_{t+1}(a_t, x_t)]). \end{aligned} \tag{8}$$

Let us now present the probabilistic boundedness result associated with the proposed control approach.

Theorem 1: Consider a discrete-time stochastic process $\{q_t\}_{t\in\mathbb{N}}$ generated by the given FID and suppose that all technical conditions from Section II are satisfied. Assume that the admissible $u_{max} \in U \subset \mathbb{R}$ and $\alpha \in [\alpha_{min}, \alpha_{max}]$ in IPC (4) satisfy

$$\mu_t u_{max}(t) \alpha_t > 0 \ \forall t \in \mathbb{N} \tag{9}$$

and the coefficients $\nu_t^b, \ \nu_t^a$ for the bid / ask quotes (6) are chosen form the following conditions:

$$e^{-k_b h_t^b} k_b \nu_t^b + e^{-k_a h_t^a} k_a \nu_t^a < \frac{2 s_t}{A_t \mu_t \alpha_t u_{max}(t)}. \tag{10}$$

Then the Markov chain $\{q_t\}$ generated by FID is locally asymptotically bounded

$$\limsup_{t\to\infty} |E[q_t]| \leq \limsup_{t\to\infty} \left| \frac{\Phi_t}{\Gamma_t} \right|, \tag{11}$$

where

$$\Phi_t := A_t\mu_t\big[\big(e^{-k_b h_t^b} - e^{-k_a h_t^a}\big) - \frac{u_{max}(t)}{s_t}(\beta_t OFI_t + \gamma_t QR_t + \delta_t TX_t)\big(e^{-k_b h_t^b} k_b \nu_t^b + e^{-k_a h_t^a} k_a \nu_t^a\big)\big] \tag{12}$$

is the market drift and

$$\Gamma_t := A_t\mu_t\big[\frac{u_{max}(t)\alpha_t}{s_t}\big(e^{-k_b h_t^b} k_b\nu_t^b + e^{-k_a h_t^a} k_a \nu_t^a\big)\big]. \tag{13}$$

is an effective gain.

Proof. Since Z_{t+1}^b and $Z_{t+1}^a(a_t, x_t)$ are Bernoulli processes, we can rewrite (8) as follows:

$$E[\Delta q_{t+1} \mid q_t] = A_t e^{-k_b b_t}\mu_t - A_t e^{-k_a a_t}\mu_t = A_t\mu_t(e^{-k_b b_t} - e^{-k_a a_t}). \tag{14}$$

Here $\Delta q_{t+1} := q_{t+1} - q_t$. Linearization of (14) around the operating point $q^* = 0$ leads to the following relation

$$E[\Delta q_{t+1} \mid q_t] \cong A_t\mu_t\big((e^{-k_b h_t^b} - e^{-k_b h_t^b} k_b \nu_t^b u_t) - (e^{-k_a h_t^a} + e^{-k_a h_t^a} k_a \nu_t^a u_t)\big), \tag{15}$$

where $h_t^b := mid_t - Sp_t/2$, and $h_t^a := mid_t + Sp_t/2$. Moreover, in a neighbourhood of u_0 we also have

$$u_t \cong \frac{u_{max}(t)}{s_t} Pr_t \tag{16}$$

From (15) and (16) we finally deduce

$$E[q_{t+1} \mid q_t] \cong q_t + \Phi_t - \Gamma_t q_t = \Phi_t + (1 - \Gamma_t)q_t, \tag{17}$$

where the market drift Φ_t and effective gain Γ_t are defined in (12) and (13), respectively. We now assume that condition (9) for $u_{max}(t)$ and α_t is satisfied. Then we conclude that $\Gamma_t > 0$ for all $t \in \mathbb{N}$. Positivity of the effective gain Γ_t and condition (10) lead to the following bounds for the effective gain Γ_t:

$$0 < \Gamma_t < 2 \ \forall t \in \mathbb{N}. \tag{18}$$

From (18) we finally obtain the classic asymptotic boundedness condition (11) for dynamic system (17) (see [17], [18]). The proof is completed. □

The system of inequalities (9)-(10) from Theorem 1 can be extended to ensure a zero market drift $\Phi_t = 0$ in equation (17). As a consequence, we obtain the corresponding asymptotic stability result for this specific case.

Corollary 1: Assume that all the conditions of Theorem 1 are satisfied and moreover

$$\alpha_{max} < \sup_{t\in\mathbb{N}} \frac{2C_t}{A_t\mu_t\big(e^{-k_b h_t^b} - e^{-k_a h_t^a}\big)}. \tag{19}$$

where $C_t := (\beta_t OFI_t + \gamma_t QR_t + \delta_t TX_t)$. Let additionally the control parameter $u_{max}(t)$ in IPC (4) is chosen as follows

$$u_{max}(t) = \frac{\big(e^{-k_b h_t^b} - e^{-k_a h_t^a}\big)}{C_t\big(e^{-k_b h_t^b} k_b \nu_t^b + e^{-k_a h_t^a} k_a \nu_t^a\big)} \ \forall t \in \mathbb{N}. \tag{20}$$

Then the equilibrium point $q^* = 0$ of the Markov chain $\{q_t\}$ generated by FID is locally asymptotically stable in the mean

$$\lim_{t\to\infty} E[q_t] = 0. \tag{21}$$

Moreover, $\{q_t\}$ converges in probability to $q^* = 0$:

$$\forall \epsilon > 0, \quad \lim_{t\to\infty} P(|q_t| \geq \epsilon) = 0. \tag{22}$$

Proof. Inequality (19) implies that

$$\alpha_t < \frac{2C_t}{A_t\mu_t\big(e^{-k_b h_t^b} - e^{-k_a h_t^a}\big)}$$

for all $t \in \mathbb{N}$. As a consequence we obtain the consistency of the inequality conditions (9)-(10) and (20). From condition (20) we deduce that $\Phi_t = 0$ in equation (17). The asymptotic stability in mean, namely, (21) immediately follows from the results of [17].

Since $Z_t^b, \ \ Z_t^a \in \{0, 1\}$ are Bernoulli processes and $Y_t^b, \ \ Y_t^a$ are stochastic processes with uniformly bounded moments, the variance $Var(q_t)$ of inventory q_t is also bounded $sup_{t\in\mathbb{N}} Var(q_t) \leq S < \infty$, with a constant $S > 0$. The convergence (stability) in probability, namely, condition (22) now follows from the celebrated Chebychev Theorem (see e.g., [19]). The proof is completed. □

Theorem 1 and Corollary 1 constitute the specific versions of the local asymptotic boundedness and local asymptotic stability results from stochastic control. We refer to [17], [18] for the necessary technical details and related facts.

Under additional conditions, specifically by assuming (19)–(20), one can derive the local asymptotic stability in the mean for the discrete-time stochastic process $\{q_t\}_{t\in\mathbb{N}}$ generated by the FID. Corollary 1 also shows that the system of inequalities governing the stability region, namely, inequalities (9)-(10) and (19)–(20) is consistent, meaning a valid solution exists even when the market drift is zero. By specifically selecting the control parameter $u_{max}(t)$ according to the rule in equation (20), the market drift Φ_t is effectively canceled out. Moreover, condition (22) in Corollary 1 implies that the inventory process q_t remains in a bounded neighborhood of $q^* = 0$ with a high probability. Let us also note that the results obtained in this section, namely Theorem 1 and Corollary 1, possess a local character. This fact is an immediate consequence of the linearization procedure applied around the equilibrium $q^* = 0$.

IV. Optimal Strategy for High-Frequency Market Making

The IPC market-making strategy, equipped with specific control parameters and bid/ask quote coefficients that satisfy inequalities (9)–(10) and (19)–(20), implies the asymptotic stability in the mean of the inventory. This stability property of the resulting IPC constitutes a highly suitable strategy for hedging purposes. On the other hand, the conditions (9)-(10) and (19)-(20) for asymptotic stability do not impose a "rigid strategy"; instead, they establish a stability framework that preserves the opportunity for Profit and Loss (PnL) based optimization. In this section, we extend the asymptotically stable IPC market-making strategy by proposing an integrated PnL and risk-aware optimization approach. We use the so-called Mark-to-Market PnL as the primary metric to evaluate the effectiveness of the proposed IPC feedback control and consider the incremental change in this PnL at each step

$$\Delta PnL_t = q_t \cdot (mid_{t+1} - mid_t). \tag{23}$$

To find optimal controls $\{\alpha_0^{opt}, \alpha_1^{opt}, \dots, \alpha_{T-1}^{opt}\}$ for the proposed IPC strategy, we now consider the following minimization problem associated with the given FID:

$$\begin{aligned} &\underset{\alpha_0,\alpha_1,\dots,\alpha_{T-1}}{\text{maximize}} && J(\alpha) \\ &\text{subject to} && (2),\ (9)\text{–}(10),\ (19)\text{–}(20) \\ & && \alpha_t \in [\alpha_{\min}, \alpha_{\max}]\ \forall t = 0, ..., T-1. \end{aligned} \tag{24}$$

Here

$$J(\alpha) := E\big[\sum_{t=0}^{T-1} (\Delta PnL_{t+1}(q_{t+1}(\alpha_t)) - \xi q_t^2) - \xi q_T^2\big]$$

is the PnL / risk based objective functional. We use here the following notation $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_{T-1})^T$. The term $\xi q_t^2,\ \xi > 0$ in $J(\alpha)$ expresses the conventional quadratic risk.

The main optimization problem (24) is formulated as finding an optimal control parameter sequence $\{\alpha_t^{opt}\}_{t=0,\dots,T-1}$ that maximizes the objective functional $J(\cdot)$. Evidently, the inventory $q_{t+1} = q_{t+1}(\alpha_t)$ is a controlled stochastic process. Its evolution is governed by the FID, where the control variable α_t modulates the probability of order execution, thereby directly influencing the trajectory of q_t. Consequently, the maximization of $J(\cdot)$ requires a balance between instantaneous profitability and the long-term stabilization of the state q_t. The optimization approach (24) establishes a dual-purpose framework for the IPC strategy. The inequality constraints derived from Theorem 1 and Corollary 1 function as a rigorous stability boundary, guaranteeing that the inventory q_t remains stochastically bounded and converges toward the equilibrium. Within this "safe operational region", the maximization of $J(\alpha)$ drives the strategy toward an optimal balance: it maximizes the expected PnL while simultaneously minimizing the associated inventory risk.

The solution procedure of the obtained optimization problem (24) is based on the Dynamic Programming (DP) principle, which characterizes an optimal control parameter sequence $\{\alpha_t^{opt}\}_{t=0,\dots,T-1},\ \alpha_t^{opt} \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of admissible control parameters α_t in problem (24). Taking into consideration the additive structure of functional $J(\cdot)$ over the given discrete-time horizon under consideration, we define the state-dependent reward

$$R(q_t, \alpha_t) := \mathbb{E}\left[\Delta PnL_{t+1}(q_{t+1}(\alpha_t)) - \xi q_t^2\right],$$

where $t = 0, ..., T-1$. Let us now introduce the value function $V(t, q)$ associated with the optimization problem (24)

$$V(t, q_t) := \max_{\alpha_t,\dots,\alpha_{T-1} \in \mathcal{A}} \mathbb{E}\left[\sum_{k=t}^{T-1} R(q_k, \alpha_k) - \xi q_T^2 \mid q_t\right].$$

where $t = 0, ..., T-1$. Evidently, the value function $V(\cdot,\cdot)$ in (25) represents the maximum expected cumulative reward from time t to the time horizon T. For any $t < T$, the optimal control parameter α_t^{opt} is determined by the fundamental discrete-time Bellman equation

$$\begin{aligned} &V(t, q_t) = \max_{\alpha_t \in \mathcal{A}} \{R(q_t, \alpha_t) + \mathbb{E}\left[V(t+1, q_{t+1}) \mid q_t\right]\}, \\ &\forall t = 0, ..., T-1 \end{aligned} \tag{25}$$

with the corresponding boundary condition

$$V(T, q_T) = -\xi q_T^2. \tag{26}$$

We refer to [1], [5] for the necessary theoretical concepts and facts. Let us now formulate the main result regarding the proposed PnL/risk-based optimal IPC strategy.

Theorem 2: Consider the IPC strategy governed by the optimal control sequence $\{\alpha_t^{opt}\}_{t=0,\dots,T-1}$ derived from the boundary-value problem (25)-(26) associated with the optimization problem (24). This optimal IPC strategy ensures a non-negative expected total reward

$$V(0,0) = \mathbb{E}[\sum_{t=0}^{T-1} R(q_t, \alpha_t^{opt})] \geq 0. \tag{27}$$

Proof. Since every α_t^{opt} from an optimal sequence $\{\alpha_t^{opt}\}_{t=0,\dots,T-1}$ belongs to the set $\mathcal{A}$, the corresponding optimal dynamics of the inventory mean is given by (17). From results of [3] it follows the existence of a feedback control with an admissible $\hat{\alpha} \in \mathcal{A}$ such that the expected gain is strictly positive

$$\mathbb{E}[\sum_{t=0}^{T-1} \Delta PnL_{t+1}(q_{t+1}(\hat{\alpha}))] > 0. \tag{28}$$

Definition of the value function $V(t, q_t)$ and inequality (28) imply the following relation

$$V(0,0) \geq \mathbb{E}[\sum_{t=0}^{T-1} \Delta PnL_{t+1}(q_{t+1}(\hat{\alpha}))] - \mathbb{E}[\sum_{t=0}^{T-1} \xi q_t^2]$$

for a sufficiently small risk penalty $\xi > 0$. We finally obtain (27). The proof is completed. □

Theorem 2 shows that the optimal IPC strategy is not only a local stabilizing controller for inventory risk, but also a locally profitability-preserving mechanism.

Corollary 2: Assume that all the conditions of Theorem 2 are satisfied. Then the resulting PnL process (23) satisfies

$$\mathbb{E}[PnL_T \mid q_0 = 0] \geq 0$$

and the terminal inventory risk is bounded $\mathbb{E}[\xi q_T^2] \leq K$, where K is a positive constant determined by the penalty coefficient ξ and the stability boundary of $\mathcal{A}$.

Corollary 2 follows immediately from Theorem 2 by setting the risk penalty $\xi > 0$ to an admissible lower bound.

To illustrate the practical implementability of the proposed approach, we conclude this section with a numerical simulation of the IPC strategy (Fig.1 and Fig.2).

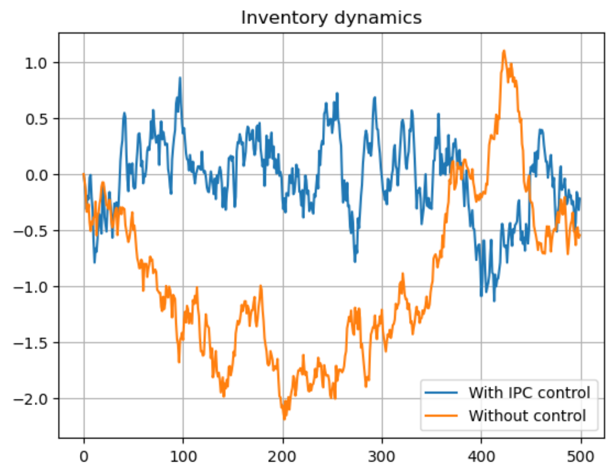

Fig. 1. Inventory dynamics for the optimal IPC strategy.

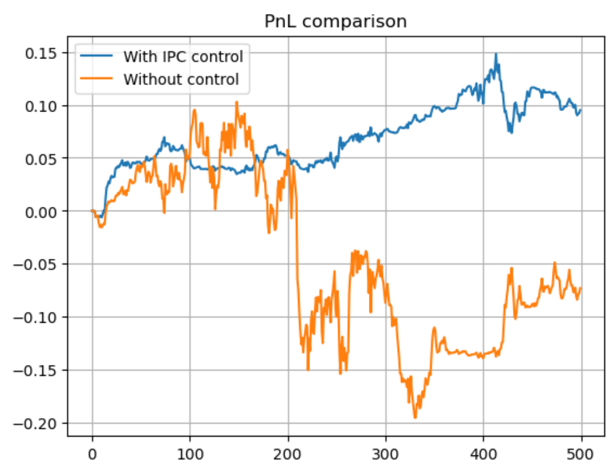

Fig. 2. PnL comparison validating the optimal IPC strategy.

As shown in Fig. 1 and 2, the feedback-driven approach effectively stabilizes the inventory q_t within the admissible region $\mathcal{A}$, preventing the divergence observed in the uncontrolled case. This simulation thus validates Corollary 2, confirming the non-negative expected growth of the cumulative PnL associated with the optimal IPC strategy.

V. Concluding Remarks

In this paper, we have proposed a novel IPC framework for high-frequency market making, providing a robust alternative to traditional stochastic optimal control models. We established the local asymptotic boundedness and local stochastic stability of the IPC involved closed-loop inventory dynamics. This result shows that under specific inequality conditions related to market drift and control gain, the inventory process remains within a bounded neighborhood of a target with high probability. This stabile behaviour is critical for the HFT market making, as it effectively damps inventory risk and prevents the divergence of positions.

Furthermore, we demonstrated that the obtained stability framework does not restrict the strategy's performance but rather defines a "safe" operational region for a PnL-risk-aware optimization. By applying the Dynamic Programming principle and deriving the celebrated Bellman equation, we developed a theoretical concept that balances profit maximization with risk awareness for locally stable inventory dynamics. The proposed optimal IPC market-making algorithm is also compatible with generic price prediction and trend-following techniques (see, e.g., [16]).

Finally note that the presented IPC approach needs to be extended by a large-scale empirical application of the proposed optimization procedure using the real-world high-frequency limit order book data.

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