Event-Triggered Stochastic Synchronization of Reaction-Diffusion Neural Networks with Deception Attacks

S Keerthana[1] and A Manivannan[1*]

Abstract— **In this paper, the problem of synchronization of reaction diffusion neural network under deception attack is investigated via event-triggered control. The attack strategy is assumed to follow a Bernoullit process, where $Pr\{\alpha = 1\} = \bar{\alpha}$. Assuming an energy constraint $|d(\ell(t, \varsigma))| \leq |\mathcal{D}\ell(t, \varsigma)|$ on the attack, a triggering mechanism and a Lyapunov Krasovskii functional are constructed to solve the optimization problem. Moreover, a set of linear matrix inequalities is developed to gurarantee that mean-square asymptotic synchronization is achieved, and the efficacy of the proposed approach is illustrated in numerical examples.**

Index Terms— **neural network, Deception attack, reaction-diffusion term, event-trigger, synchronization**

I. INTRODUCTION

Reaction–diffusion neural networks (RDNNs) have emerged as an extension of traditional neural network to better represent systems where spatial interactions is crucial. Classical neural networks which is based on ordinary differential equations (ODEs) are effective in tasks such as pattern recognition, optimization, and control but spatial distribution is not taken into account. RDNNs address this issue by incorporating diffusion terms, allowing it to capture both temporal dynamics and spatial behavior.

RDNNs incorporate spatial dynamics via partial differential equations. Here, the reaction term represents local neural interactions, whereas the diffusion term models how signals spread over space. This makes RDNNs more suitable for modeling complex systems with both spatial and temporal components [1], [2]. RDNNs can exhibit a wide variety of behaviors and synchronization, where systems adjust their states to follow the same trajectory, is particularly significant. The system plays a crucial role in various applications, including secure communication, image encryption, multi-agent coordination, and the study of neurological disorders such as epilepsy. The primary objective is to create a control input that enables a response system to track a drive system.

In practice, implementing this is not a straightforward process. Typically, sensors and controllers exchange data via networks with restricted bandwidth. In traditional time-triggered control, data is transmitted at regular time intervals, regardless of whether it is required. This results in unnecessary communication, increased energy usage, and network congestion, particularly in large-scale RDNNs.

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Event-triggered control (ETC) is employed to address these problems. In ETC, data is transmitted only when a specific condition is fulfilled, for instance, when the current state varies substantially from the previous one that was sent. The communication is reduced while still maintaining good performance. Since updates are checked only at sampling times, the Zeno problem of infinitely many events in a short time does not occur [3], [4]. As control systems become more interconnected, they also become more susceptible to cyber-physical attacks on their communication pathways. Deception attacks are particularly damaging because they modify the data being transmitted [5], [6]. In such attacks, an adversary replaces the true signal with a false one, misleading the controller and potentially destabilizing the system.

Existing studies on deception attacks primarily focus on simple systems, such as ODE. RDNNs, which are more representative of spatially distributed and security-critical systems, have garnered little attention in this context. The gap necessitates research into the impact of deception attacks on RDNNs and the creation of effective control strategies to guarantee their secure synchronization.

Notations: Throughout this paper, the following notation is adopted. The symbols $\mathbb{R}^n$ and $\mathbb{R}^m$ represent the n-dimensional and m-dimensional Euclidean spaces, respectively. For a matrix Q, the notation $Q > 0$ (respectively, $Q \geq 0$) indicates that Q is positive definite (respectively, positive semi-definite), while $Q < 0$ (respectively, $Q \leq 0$) denotes that Q is negative definite (respectively, negative semi-definite). The symbol $*$ is used to denote entries determined by symmetry in a symmetric matrix. For any matrix X, its transpose is represented by X^T. Furthermore, the notation $\mathrm{col}\{c, d\}$ denotes the column vector: $\begin{bmatrix} c \\ d \end{bmatrix}$. The diagonal matrix $\mathrm{diag}\{c, d\}$ replaces this matrix by: $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

II. PROBLEM FORMULATION

The mathematical model considered in this work involves delayed RDNNs with event-triggered mechanisms and deception attacks. Before proceeding to the main analysis, a set of fundamental assumptions required for the problem development is established.

A. Delayed Reaction-Diffusion Neural Networks

Consider the drive system (master):

$$\frac{\partial x(t,\varsigma)}{\partial t} = H\frac{\partial^2 x(t,\varsigma)}{\partial \varsigma^2} - Ax(t,\varsigma) + Bg(x(t,\varsigma)) + Cg(x(t-\eta(t),\varsigma)) + \hat{I} \tag{1}$$

with the Dirichlet boundary condition $x(t,\varsigma) = 0$ for all $(t,\varsigma) \in [-\eta,\infty) \times \partial\Omega$, together with the initial state condition $x(t,\varsigma) = \iota(t,\varsigma)$ on $[-\eta,0] \times \Omega$. Here, $\Omega = \{\varsigma \mid \beta_1 \le \varsigma_k \le \beta_2\} \subset \mathbb{R}^m$, diffusion matrix $H = \mathrm{diag}\{h_1,\dots,h_n\} > 0$, $A = \mathrm{diag}\{a_1,\dots,a_n\} > 0$, $B, C \in \mathbb{R}^{n\times n}$, and the bounded time-varying delay satisfies $0 \le \eta(t) \le \eta$ and $\dot{\eta}(t) \le \hat{\mu}$.

The response system (slave) is:

$$\frac{\partial y(t,\varsigma)}{\partial t} = H\frac{\partial^2 y(t,\varsigma)}{\partial \varsigma^2} - Ay(t,\varsigma) + Bg(y(t,\varsigma)) + Cg(y(t-\eta(t),\varsigma)) + \hat{I} + \mathfrak{u}(t,\varsigma) \tag{2}$$

Let the synchronization error be defined as $\ell(t,\varsigma) = y(t,\varsigma) - x(t,\varsigma)$. Accordingly, the associated error dynamics can be expressed as follows:

$$\frac{\partial \ell(t,\varsigma)}{\partial t} = H\frac{\partial^2 \ell(t,\varsigma)}{\partial \varsigma^2} - A\ell(t,\varsigma) + Bf(\ell(t,\varsigma)) + Cf(\ell(t-\eta(t),\varsigma)) + \mathfrak{u}(t,\varsigma) \tag{3}$$

where $f(\ell(t,\varsigma)) = g(y(t,\varsigma)) - g(x(t,\varsigma))$.

B. Event-Triggered Control Under Deception Attack

The sensor samples at $t = kh$ with period $h > 0$. For $t \in [t_k h, t_{k+1}h)$, there exists $j \ge 0$ such that $t \in [t_k h + jh, t_k h + (j+1)h)$. Define $\phi(t) = t - (t_k h + jh)$, $0 \le \phi(t) \le h$, and the error:

$$e(t-\phi(t),\varsigma) = \ell(t-\phi(t),\varsigma) - \ell(t_k h,\varsigma) \tag{4}$$

The deception attack on the sensor-controller channel is modeled as:

$$\hat{z}(t_k h,\varsigma) = (1-\alpha(t))z(t_k h,\varsigma) + \alpha(t)d(z(t_k h,\varsigma)) \tag{5}$$

where $\alpha(t)$ is assumed to obey a Bernoulli distribution characterized by

$$\Pr\{\alpha(t)=1\} = \bar{\alpha}, \qquad \Pr\{\alpha(t)=0\} = 1-\bar{\alpha}.$$

. The attack signal satisfies the energy constraint: $\|d(\ell(t,\varsigma))\| \le \|\mathcal{D}\ell(t,\varsigma)\|, \quad \mathcal{D} = \mathrm{diag}\{d_1,\dots,d_n\}$.

The attack-aware event-triggered condition:

$$e^T(t-\phi(t),\varsigma)\psi e(t-\phi(t),\varsigma) \le \sigma \hat{z}^T(t_k h,\varsigma)\psi\hat{z}(t_k h,\varsigma) \tag{6}$$

The control law:

$$\mathfrak{u}(t,\varsigma) = K\hat{z}(t_k h,\varsigma), \quad t \in [t_k h, t_{k+1}h) \tag{7}$$

Substituting (5) and (4) into (7):

$$\mathfrak{u}(t,\varsigma) = K(1-\alpha_k)\big(\ell(t-\phi(t),\varsigma) - e(t-\phi(t),\varsigma)\big) + K\alpha_k d\big(\ell(t-\phi(t),\varsigma) - e(t-\phi(t),\varsigma)\big) \tag{8}$$

where $\alpha_k = \alpha(t_k h)$. The error dynamics now become:

$$\frac{\partial \ell(t,\varsigma)}{\partial t} = H\frac{\partial^2 \ell(t,\varsigma)}{\partial \varsigma^2} - A\ell(t,\varsigma) + Bf(\ell(t,\varsigma)) + Cf(\ell(t-\eta(t),\varsigma)) + K(1-\alpha_k)\big(\ell(t-\phi(t),\varsigma) - e(t-\phi(t),\varsigma)\big) + K\alpha_k d\big(\ell(t-\phi(t),\varsigma) - e(t-\phi(t),\varsigma)\big) \tag{9}$$

For simplicity, define $r_b = [0_{n\times(b-1)n}, I_n, 0_{n\times(13-b)n}]$, $b = 1,\dots,13$.

C. Assumption and Lemma

Assumption 2.1: For any $\nu_1,\nu_2 \in \mathbb{R}$ with $\nu_1 \ne \nu_2$, the activation function satisfies

$$\rho_i^- \le \frac{f_i(\nu_1) - f_i(\nu_2)}{\nu_1 - \nu_2} \le \rho_i^+.$$

$F_1 = \mathrm{diag}\{\rho_1^-\rho_1^+,\dots,\rho_n^-\rho_n^+\}$ and $F_2 = \mathrm{diag}\Big\{\frac{\rho_1^-+\rho_1^+}{2},\dots,\frac{\rho_n^-+\rho_n^+}{2}\Big\}$.

Lemma 2.1: Assume that $\phi : [a,b] \to \mathbb{R}^m$ is a continuously differentiable function. Then, for any positive definite matrix P and arbitrary matrix $\mathcal{Y}$, the inequality given below is satisfied:

$$-\int_a^b \dot{\phi}^T(t)P\dot{\phi}(t)\,dt \le \chi^T(t)\Big[\ell\,\mathcal{Y}^T P^{-1}\mathcal{Y}\Big]\chi(t) - 2\chi^T(t)\Big\{\mathcal{Y}\big(\phi(b)-\phi(a)\big)\Big\}$$

where

$$\chi(t) = \begin{bmatrix}\phi^T(b) & \phi^T(a)\end{bmatrix}^T, \qquad \ell = b-a.$$

III. Main Results

This section derives adequate criteria to guarantee synchronization between the delay-dependent drive system (1) and the response system (2) . The investigation is conducted by constructing a suitable LKF within an event-triggered transmission framework subject to deception attacks. The resulting criteria are then expressed in the form of LMI, culminating in the theorem stated below.

Theorem 3.1: For given positive scalars η, h, $\hat{\mu}$, σ, γ, attack probability $\bar{\alpha} \in [0,1]$, and attack bounded matrix $\mathcal{D}$, the error system under deception attack is mean-square asymptotically stable via the ETC with gain $K = Z^{-1}L$, if there exist symmetric positive definite matrices P, $ZD = (ZD)^T$ Q_1, Q_3, R_1, R_2, R_3, S, U_1, U_2, ψ, positive diagonal matrices Q_2, Λ_1, Λ_2, Λ_3, and any appropriate dimension matrix L, such that the LMI holds.

$$\Xi = \begin{bmatrix}\Gamma & \eta M \\ * & -R_2\end{bmatrix} \tag{10}$$

$$
\begin{aligned}
\Gamma = {} & r_1^T Q_1 r_1 + r_1^T R_1 r_1 + \eta^2 r_1^T U_1 r_1 \\
& + \frac{\eta^4}{4} r_1^T U_2 r_1 - \frac{\pi^2}{4} r_1^T S r_1 - r_1^T F_1 \Lambda_1 r_1 - 2 r_1^T Z A r_1 \\
& - 2\frac{\pi^2}{(\upsilon_2-\upsilon_1)^2} r_4^T Z D r_4 + \bar{\alpha} r_1^T \mathcal{D}^T \Lambda_3 \mathcal{D} r_1 + r_1^T Q_2 r_{10} \\
& + r_1^T F_2 \Lambda_1 r_5 + r_1^T Z B r_5 + r_1^T Z C r_6 + \frac{\pi^2}{4} r_1^T S r_7 \\
& + r_1^T L(1-\bar{\alpha}) r_7 - r_1^T L(1-\bar{\alpha}) r_9 + r_1^T P r_{10} - \gamma r_1^T \ell^T A r_{10} \\
& - r_1^T Z r_{10} - r_2^T F_1 \Lambda_2 r_2 - (1-\hat{\mu}) r_2^T R_1 r_2 + r_2^T F_2 \Lambda_2 r_6 \\
& - r_3^T Q_1 r_3 - r_3^T Q_2 r_8 - 2 r_4^T D Z r_4 + r_5^T R_3 r_5 - r_5^T \Lambda_1 r_5 \\
& + \gamma r_5^T \ell^T B r_{10} - (1-\hat{\mu}) r_6^T R_3 r_6 - r_6^T \Lambda_2 r_6 + \gamma r_6^T \ell^T l r_{10} \\
& - \frac{\pi^2}{4} r_7^T S r_7 + \bar{\alpha}\sigma r_7^T \mathcal{D}^T \psi \mathcal{D} r_7 - \sigma(1-\bar{\alpha}) r_7^T \psi r_9 \\
& - \bar{\alpha}\sigma r_7^T \mathcal{D}^T \psi \mathcal{D} r_9 + \gamma r_7^T L^T r_{10} + r_8^T Q_3 r_8 - r_8^T 2\eta M r_8 \\
& + r_8^T \eta M r_{10} + \sigma(1-\bar{\alpha}) r_9^T \psi r_9 - r_9^T \psi r_9 + \bar{\alpha}\sigma r_9^T \mathcal{D}^T \psi \mathcal{D} r_9 \\
& - \gamma r_9^T L^T r_{10} - h^2 r_{10}^T S r_{10} - r_{10}^T 2\eta M r_{10} + -2\gamma r_{10}^T Z r_{10} \\
& + r_{10}^T (\eta)^2 R_2 r_{10} + r_{10}^T Q_3 r_{10} - r_{11}^T U_1 r_{11} - 3 r_{11}^T U_1 r_{11} \\
& - \frac{6}{\eta} r_{11}^T U_1 r_{12} - \frac{12}{\eta^2} r_{12}^T U_1 r_{12} - r_{12}^T U_2 r_{12} - 2 r_{12}^T U_2 r_{12} \\
& + \frac{6}{\eta} r_{12}^T U_2 r_{13} - \frac{18}{\eta^2} r_{13}^T U_2 r_{13}.
\end{aligned}
$$

Proof: The following LKF candidate is constructed as:

$$
\mathcal{V}(\ell(t,\varsigma)) = \sum_{i=1}^{7} \mathcal{V}_i(\ell(t,\varsigma)) \tag{11}
$$

$$
\begin{aligned}
\mathcal{V}_1(\ell(t,\varsigma)) &= \int_\Omega \ell^T(t,\varsigma)\ P\ \ell(t,\varsigma)\, d\varsigma \\
\mathcal{V}_2(\ell(t,\varsigma)) &= \int_\Omega \ell_r^T(t,\varsigma)\ \gamma Z D\ \ell_r(t,\varsigma)\, d\varsigma \\
\mathcal{V}_3(\ell(t,\varsigma)) &= \int_{t-\eta}^{t} \begin{bmatrix} \ell(s,\varsigma) \\ \dot{\ell}(s,\varsigma) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} \ell(s,\varsigma) \\ \dot{\ell}(s,\varsigma) \end{bmatrix} ds \\
\mathcal{V}_4(\ell(t,\varsigma)) &= \int_{t-\eta(t)}^{t} \ell^T(s,\varsigma) R_1 \ell(s,\varsigma)\, ds \\
&\quad + \eta \int_{-\eta}^{0}\int_{t+\varkappa}^{t} \dot{\ell}^T(s,\varsigma) R_2 \dot{\ell}(s,\varsigma)\, ds\, d\varkappa \\
&\quad + \int_{t-\eta(t)}^{t} f^T\big(\ell(s,\varsigma)\big) R_3 f\big(\ell(s,\varsigma)\big)\, ds\ d\varsigma
\end{aligned}
$$

The discontinuous Lyapunov Functional is of the form,

$$
\begin{aligned}
\mathcal{V}_5(\ell(t,\varsigma)) = \int_\Omega \Bigg\{ & h^2 \int_{t-\phi(t)}^{t} \dot{\ell}^T(u,\varsigma)\ S\ \dot{\ell}(u,\varsigma)\, du \\
& - \frac{\pi^2}{4} \int_{t-\phi(t)}^{t} \big[\ell(u,\varsigma) - \ell(t-\phi(t),\varsigma)\big]^T S \\
& \big[\ell(u,\varsigma) - \ell(t-\phi(t),r)\big]\, du \Bigg\} d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_6(\ell(t,\varsigma)) &= \int_\Omega \left\{ \eta \int_{t-\eta}^{t}\int_{\varkappa}^{t} \ell^T(u,\varsigma)\ U_1\ \ell(u,\varsigma)\, du\, d\varkappa \right\} d\varsigma \\
\mathcal{V}_7(\ell(t,\varsigma)) &= \int_\Omega \left\{ \frac{\eta^2}{2} \int_{t-\eta}^{t}\int_{\varkappa}^{t}\int_{\beta}^{t} \ell^T(u,\varsigma)\ U_2\ \ell(u,\varsigma)\, du\, d\beta\, d\varkappa \right\} d\varsigma
\end{aligned}
$$

Differentiating (11), one can obtain

$$
\dot{\mathcal{V}}(\ell(t,\varsigma)) = \sum_{i=1}^{7} \dot{\mathcal{V}}_i(\ell(t,\varsigma)) \tag{12}
$$

where

$$
\begin{aligned}
\dot{\mathcal{V}}_1(\ell(t,\varsigma)) &= \int_\Omega 2\ell^T(t,\varsigma)\ P\ \dot{\ell}(t,\varsigma)\, d\varsigma \\
\dot{\mathcal{V}}_2(\ell(t,\varsigma)) &= \int_\Omega 2\ell_r^T(t,\varsigma)\ \gamma Z D\ \ell_{rt}(t,\varsigma)\, d\varsigma \\
\dot{\mathcal{V}}_3(\ell(t,\varsigma)) &= \ell^T(t,\varsigma) Q_1 \ell(t,\varsigma) + \ell^T(t,\varsigma) Q_2 \dot{\ell}(t,\varsigma) \\
&\quad + \dot{\ell}^T(t,\varsigma) Q_3 \dot{\ell}(t,\varsigma) - \ell^T(t-\eta,\varsigma) Q_1 \ell(t-\eta,\varsigma) \\
&\quad - \ell^T(t-\eta,\varsigma) Q_2 \dot{\ell}(t-\eta,\varsigma) \\
&\quad - \dot{\ell}^T(t-\eta,\varsigma) Q_3 \dot{\ell}(t-\eta,\varsigma) \\
\dot{\mathcal{V}}_4(\ell(t,\varsigma)) &= \int_\Omega \ell^T(t,\varsigma) R_1 \ell(t,\varsigma) + \eta^2 \dot{\ell}^T(t,\varsigma) R_2 \dot{\ell}(t,\varsigma) \\
&\quad + -\eta \int_{t-\eta}^{t} \dot{\ell}^T(s,\varsigma)\ R_2\ \dot{\ell}(s,\varsigma)\, ds \\
&\quad + f^T(\ell(t,\varsigma)) R_3 f(\ell(t,\varsigma)) \\
&\quad - (1-\hat{\mu})\, f^T(\ell(t-\eta(t),\varsigma)) R_3 f(\ell(t-\eta(t),\varsigma))\, d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathcal{V}}_5(\ell(t,\varsigma)) &= \int_\Omega \Bigg\{ h^2 \dot{\ell}^T(t,\varsigma)\ S\ \dot{\ell}(t,\varsigma) - \frac{\pi^2}{4}\big[\ell(t,\varsigma) \\
&\quad - \ell(t-\phi(t),\varsigma)\big]^T S \big[\ell(t,\varsigma) - \ell(t-\phi(t),\varsigma)\big] \Bigg\} d\varsigma \\
\dot{\mathcal{V}}_6(\ell(t,\varsigma)) &= \int_\Omega \Bigg\{ \eta^2 \ell^T(t,\varsigma)\ U_1\ \ell(t,\varsigma) \\
&\quad - \eta \int_{t-\eta}^{t} \ell^T(s,\varsigma)\ U_1\ \ell(s,\varsigma)\, ds \Bigg\} d\varsigma \\
\dot{\mathcal{V}}_7(\ell(t,\varsigma)) &= \int_\Omega \Bigg\{ \frac{\eta^4}{4} \ell^T(t,\varsigma)\ U_2\ \ell(t,\varsigma) \\
&\quad - \frac{\eta^2}{2} \int_{t-\eta}^{t}\int_{\varkappa}^{t} \ell^T(s,\varsigma)\ U_2\ \ell(s,\varsigma)\, ds\, d\varkappa \Bigg\} d\varsigma
\end{aligned}
$$

From lemma (2.1), one have,

$$-\int_{t-\eta}^{t}\dot{\ell}^T(s,\varsigma)R_2\dot{\ell}(s,\varsigma)\,ds \le \eta^2\xi^T(t)M^TR_2^{-1}M\xi(t)$$
$$-2\eta\xi^T(t)M\left(\dot{\ell}(t,\varsigma)-\dot{\ell}(t-\eta,\varsigma)\right) \tag{13}$$

By applying Corollary 4 in [7], the following estimation for the single-integral term can be obtained:

$$\begin{aligned}
&-\eta\int_{t-\eta}^{t}\ell^T(s,\varsigma)\;U_1\;\ell(s,\varsigma)\,ds\\
&\le-\left(\int_{t-\eta}^{t}\ell(s,\varsigma)\,ds\right)^T U_1\left(\int_{t-\eta}^{t}\ell(s,\varsigma)\,ds\right)\\
&\quad-3\bigg\{\left[\int_{t-\eta}^{t}\ell(s,\varsigma)\,ds-\frac{2}{\eta}\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\right]^T\\
&\quad U_1\left[\int_{t-\eta}^{t}\ell(s,\varsigma)\,ds-\frac{2}{\eta}\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\right]\bigg\}
\end{aligned} \tag{14}$$

By applying corollary 4 in [8], the following estimation for the double integral term can be obtained:

$$\begin{aligned}
&-\frac{\eta^2}{2}\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell^T(s,\varsigma)\,U_2\,\ell(s,\varsigma)\,ds\,d\varkappa\\
&\le-\left(\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\right)^T U_2\\
&\quad\left(\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\right)-2\\
&\bigg\{\bigg[-\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\\
&+\frac{3}{\eta}\int_{t-\eta}^{t}\int_{\varkappa}^{t}\int_{\beta}^{t}\ell(s,\varsigma)\,ds\,d\beta\,d\varkappa\bigg]^T\\
&\times U_2\bigg[-\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell(s,\varsigma)\,ds\,d\varkappa\\
&+\frac{3}{\eta}\int_{t-\eta}^{t}\int_{\varkappa}^{t}\int_{\beta}^{t}\ell(s,\varsigma)\,ds\,d\beta\,d\varkappa\bigg]\bigg\}
\end{aligned} \tag{15}$$

From Assumption 2.1, for any $\Lambda_1 > 0$, this holds:

$$0 \le \xi^T(t,\varsigma)\begin{bmatrix}l_1\\ l_5\end{bmatrix}^T\begin{bmatrix}-F_1\Lambda_1 & F_2\Lambda_1\\ * & -\Lambda_1\end{bmatrix}\begin{bmatrix}l_1\\ l_5\end{bmatrix}\xi(t,\varsigma) \tag{16}$$

In a similar manner, for any diagonal matrix $\Lambda_2 > 0$ with compatible dimensions, the following relation holds:

$$0 \le \xi^T(t,\varsigma)\begin{bmatrix}l_2\\ l_6\end{bmatrix}^T\begin{bmatrix}-F_1\Lambda_2 & F_2\Lambda_2\\ * & -\Lambda_2\end{bmatrix}\begin{bmatrix}l_2\\ l_6\end{bmatrix}\xi(t,\varsigma) \tag{17}$$

For the error system (9) subjected to deception attacks, the following zero equality can be established:

$$\begin{aligned}
0=2&\int_{\Omega}\left[\ell^T(t,\varsigma)+\gamma\dot{\ell}^T(t,\varsigma)\right]Z\Big(-\dot{\ell}(t,\varsigma)+\\
&H\frac{\partial^2\ell(t,\varsigma)}{\partial\varsigma^2}-A\,\ell(t,\varsigma)+B\,f(\ell(t,\varsigma))\\
&+C\,f\big(\ell(t-\eta(t),\varsigma)\big)+K(1-\alpha_k)\ell(t-\phi(t),\varsigma)\\
&-K(1-\alpha_k)e(t-\phi(t),\varsigma)+K\alpha_k d\big(\ell(t-\phi(t),\varsigma)\\
&-e(t-\phi(t),\varsigma)\big)\Big)\,d\varsigma
\end{aligned} \tag{18}$$

For any positive diagonal matrix $\Lambda_3 > 0$, the attack energy constraint $\|d(\ell(t_kh,\varsigma))\| \le \|\mathcal{D}\ell(t_kh,\varsigma)\|$ implies:

$$\begin{aligned}
0 \le\;& \bar{\alpha}\ell^T(t-\phi(t),\varsigma)\mathcal{D}^T\Lambda_3\mathcal{D}\ell(t-\phi(t),\varsigma)\\
&-\bar{\alpha}d^T(\ell(t-\phi(t),\varsigma))\Lambda_3 d(\ell(t-\phi(t),\varsigma))
\end{aligned}$$

Taking mathematical expectation of the event-triggered condition (6) and using the upper bound $d^T\psi d \le \ell^T\mathcal{D}^T\psi\mathcal{D}z$:

$$\begin{aligned}
0 \le\;& \sigma(1-\bar{\alpha})\ell^T(t_kh,\varsigma)\psi\ell(t_kh,\varsigma)+\bar{\alpha}\sigma\ell^T(t_kh,\varsigma)\\
&\mathcal{D}^T\psi\mathcal{D}\ell(t_kh,\varsigma)-e^T(t_kh,\varsigma)\psi e(t_kh,\varsigma)
\end{aligned} \tag{19}$$

By employing integration by parts together with the Dirichlet boundary condition, the following result is derived:

$$\begin{aligned}
&\int_{\Omega}\left[\ell^T(t,\varsigma)+\gamma\frac{\partial\ell^T(t,\varsigma)}{\partial\varsigma}\right]H\frac{\partial^2\ell(t,\varsigma)}{\partial\varsigma^2}\,d\varsigma\\
&\qquad=-2\int_{\Omega}H\frac{\partial\ell^T(t,\varsigma)}{\partial\varsigma}Z\frac{\partial\ell(t,\varsigma)}{\partial\varsigma}\,d\varsigma\\
&\qquad\quad+\gamma\int_{\Omega}H\frac{\partial\dot{\ell}^T(t,\varsigma)}{\partial\varsigma}Z\frac{\partial\ell(t,\varsigma)}{\partial\varsigma}\,d\varsigma
\end{aligned} \tag{20}$$

Combining (12) to (20), and taking mathematical expectation $\mathbb{E}[\cdot]$, one can obtain:

$$\mathbb{E}\left\{\frac{\partial}{\partial t}\mathcal{V}(\ell(t,\varsigma))\right\} \le \int_{\Omega}\xi^T(s,\varsigma)\;\Gamma\;\xi(s,\varsigma)\,d\varsigma \tag{21}$$

The augmented vector $\xi(t,\varsigma)$ is defined as:

$$\begin{aligned}
\xi^T(t,\varsigma)=\Big[&\ell^T(t,\varsigma)\;\ell^T(t-\eta(t),\varsigma)\;\ell^T(t-\eta,\varsigma)\;\ell_r^T(t,\varsigma)\\
&f^T(z((t,\varsigma))\;f^T(z((t-\eta(t),\varsigma))\;\ell^T(t-\phi(t),\varsigma)\\
&\dot{\ell}^T(t-\eta,\varsigma)\;e^T(t-\phi(t),\varsigma)\;\dot{\ell}^T(t,\varsigma)\\
&\int_{t-\eta}^{t}\ell^T(s,\varsigma)\,ds\;\int_{t-\eta}^{t}\int_{\varkappa}^{t}\ell^T(s,\varsigma)\,ds\,d\varkappa\\
&\int_{t-\eta}^{t}\int_{\varkappa}^{t}\int_{\beta}^{t}\ell^T(s,\varsigma)\,ds\,d\beta\,d\varkappa\Big]
\end{aligned}$$

If $\Xi < 0$, then $\mathbb{E}\{\dot{\mathcal{V}}(t)\} < 0$, guaranteeing the existence of a scalar $\mu > 0$ satisfying:

$$\mathbb{E}\{\|\ell(t,\varsigma)\|^2\} \le \mathbb{E}\{\|z(0,\varsigma)\|^2\}e^{-\mu t} \tag{22}$$

Hence, the error system achieves mean-square asymptotic stability. Consequently, synchronization of the drive system (1) with the response system (2) is guaranteed under the controller (9), even in the presence of deception attacks. This brings the proof to its conclusion. ■

Remark 3.1: When $\bar{\alpha} = 0$, no deception attack occurs and the LMI condition reduces to the attack-free case. As $\bar{\alpha}$ increases, the additional term $\bar{\alpha}\mathcal{D}^T\Lambda_3\mathcal{D}$ becomes more dominant, making the LMI more restrictive. This reflects the reduced feasibility region and characterizes the maximum tolerable attack probability for which synchronization can still be guaranteed.

IV. Numerical Example

In order to validate the practical utility and performance of the event-triggered control approach introduced here, a numerical illustration involving RDNNs under deception attacks is examined in this section.

Consider the following RDNNs defined over the spatial domain $\Omega = [0,1]$ as in [5]:

$$\frac{\partial x(t,\varsigma)}{\partial t} = H\frac{\partial^2 x(t,\varsigma)}{\partial \varsigma^2} - Ax(t,\varsigma) + Bg(x(t,\varsigma)) + Cg(x(t-\eta(t),\varsigma)), \qquad (23)$$

where $x(t,\varsigma) \in \mathbb{R}^2$ denotes the state vector. The system parameters are chosen as:

$$H = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.11 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.6 & -0.7 \\ 5.1 & -2.4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.6 & -2.3 \\ 1.4 & 1.5 \end{bmatrix}.$$

The activation function is taken as $g(x(t,\varsigma)) = \tanh(x(t,\varsigma))$. Moreover, the considered time-varying delay is given by $\eta(t) = 1.4 - 0.2e^{-0.1t}$, from which it follows that $\eta = 1.4, \quad \hat{\mu} = 0.02$. In addition, the initial conditions corresponding to systems (1) and (2) are selected as $x(s,\varsigma) = \left[1.2\sin(\pi s) \ -1.6\sin(\pi s)\right]^T, \quad y(s,\varsigma) = \left[-2.5\sin(\pi s) \ 2.2\sin(\pi s)\right]^T$.

The communication channel is subject to deception attacks modeled by a Bernoulli random variable $\alpha(t)$:

$$\alpha(t) = \begin{cases} 1, & \text{attack occurs,} \\ 0, & \text{otherwise,} \end{cases} \quad \Pr\{\alpha(t) = 1\} = \bar{\alpha} = 0.5.$$

The deception signal satisfies:

$$\|d(\ell(t,\varsigma)\| \le \|\mathcal{D}\ell(t,\varsigma)\|, \quad \mathcal{D} = \mathrm{diag}\{0.15, 0.13\}$$

The threshold parameter is $\sigma = 0.1$, the gain matrix K is obtained by solving the derived LMIs: $K = \begin{bmatrix} -18.2648 & -23.0704 \\ -3.7066 & -27.8984 \end{bmatrix}$ and the triggering matrix ϕ is given by $\phi = \begin{bmatrix} 1.8885 & 0.1283 \\ 0.1283 & 1.9833 \end{bmatrix}$

Fig. (1) illustrate the synchronization of errors model. Fig. (2) gives the synchronization of errors model at specific timeinstants. Fig. (3), tells about the Synchronization Error Convergence and Bernoulli Deception Attack Signal of the proposed method.

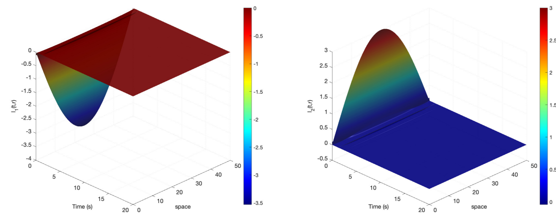

Fig. 1. Synchronization of Error

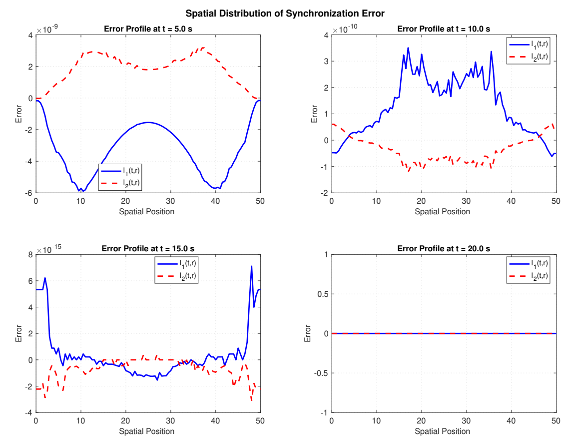

Fig. 2. Synchronization of Error State at specific timestamp

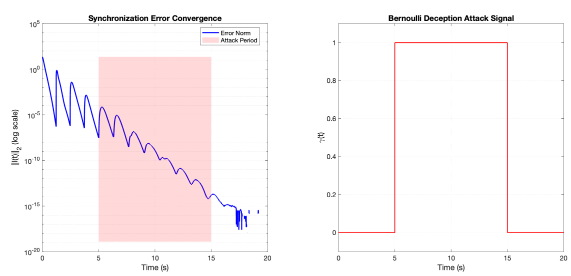

Fig. 3. Event-Triggered Synchronization under Deception Attack

V. Conclusion

This paper investigated event-triggered synchronization of RDNNs under deception attacks. A Bernoulli-distributed attack model with energy constraint was proposed, and an attack-aware event-triggered condition was developed. LMI-based stability criteria were derived to ensure mean-square asymptotic stability. Future work will focus on adaptive event-triggered mechanisms and finite-time synchronization under hybrid cyber attacks.

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