Routh Matrix Algorithm: A State-Space Approach

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***Abstract*— In contrast to the classical Routh-Hurwitz stability tests, the algorithm produces a square matrix R composed of Routh array elements. By construction, the matrix is compatible with the state vector of a state-space model. The similarity transformation R reduces a system matrix in observable canonical form to a tridiagonal form. That form is characteristic of an electrical RLC ladder network.**

I. INTRODUCTION

Mathematically, a linear state-space model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is exponentially stable if all roots of the characteristic polynomial have negative real parts. In practice, when numbers are available, an iterative program can search for the characteristic roots to check stability. A numerical search is necessary in higher order cases, because an algebraic (symbolic) solution for the roots does not exist for polynomials of a degree higher than four.

Nevertheless, Routh's criterion determines stability without the roots themselves. Routh's algorithm uses merely the coefficients of the characteristic polynomial, and the algorithm is algebraic. The result is a nonsquare array with rows of variable length. All elements in the leftmost column of the array must be positive for the model to be exponentially stable.

Subdeterminants of the Hurwitz matrix form an equivalent criterion for stability. The Hurwitz matrix is square, but it does not conform with the state vector. A third algorithm relies on a continued-fraction expansion of the ratio of the even and odd parts of the characteristic polynomial (Wall 1945) [6]. The three approaches are demonstrated in a *Schaum's Outline* edition [3]. A socalled Routh canonical form of the system matrix is linked to electrical RLC ladder networks (Yarlagadda 1968) [7]. Chen (1974) expressed a certain Lyapunov function in terms of the Routh array elements rather than the coefficients of the characteristic polynomial [2].

To recast the stability analysis in state space, the present article searches a matrix consistent with multiplication by the state vector. The key contributions of this article are the following (Sections IV and V):

1) A square Routh matrix that transforms $\mathbf{A}$ to a particular tridiagonal form; this establishes
2) an analogous electrical network; and this permits
3) a design-by-analogy approach.

For an nth order system the characteristic polynomial can be written $p(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0$, where D is the (Heaviside) differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$. Assume immediately that all coefficients are positive, which is necessary for exponential stability (Descartes' rule of signs). The leading coefficient is 1 without loss of generality – if there were an a_n, it could be divided through all other coefficients in the characteristic equation.

II. CHARACTERISTIC POLYNOMIAL

A square matrix $\mathbf{A}$ and a vector $\mathbf{v}$ may generate a Krylov sequence. Let $\mathbf{v}_1 = \mathbf{v}$ be any nonnull vector, then the Krylov sequence is defined by $\mathbf{v}_{i+1} = \mathbf{A}\mathbf{v}_i = \mathbf{A}^i\mathbf{v}_1$. There will be a first vector which is linearly dependent on the preceding ones. The preceding vectors form a basis for a subspace.

Now replace the arbitrary vector $\mathbf{v}$ by the state-space position vector $\mathbf{x}$. Then the first vector in the sequence is $\mathbf{x}$ itself $\left(\mathbf{A}^0\mathbf{x}\right)$. The second vector is $\mathbf{A}^1\mathbf{x}$, which is the velocity vector $\dot{\mathbf{x}}$. The third vector is the acceleration vector $\ddot{\mathbf{x}}$ – and so forth into higher dimensions. The juxtaposition of all the vectors is here called the *kinematic matrix*,

$$\mathbf{X} = \left[\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \overset{(n-1)}{\mathbf{x}}\right]$$

The following derivations require the inverse of the kinematic matrix, and $\operatorname{rank}\mathbf{X} = n$ is therefore assumed. If, for example, $\mathbf{x}$ happens to be an eigenvector, the Krylov subspace collapses to a single dimension. Thus, if possible, the first vector should be chosen to generate independent kinematic vectors. The vector of all ones is a candidate.

Now form the matrix $\mathbf{AX}$, which is $\mathbf{A}$ left-multiplied onto each column vector of $\mathbf{X}$. The first vector is $\dot{\mathbf{x}}$, but that is also the second vector of $\mathbf{X}$. Moving on, the second vector of $\mathbf{AX}$ is the third vector of $\mathbf{X}$. This can go on until the last vector of $\mathbf{AX}$, which must be found. Thus far the pattern is the following in matrix notation,

$$\mathbf{AX} = \mathbf{X}\begin{bmatrix} 0 & 0 & 0 & 0 & ? \\ 1 & 0 & 0 & 0 & ? \\ 0 & 1 & 0 & 0 & ? \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & ? \end{bmatrix}$$

The subdiagonal of ones shifts the vectors in $\mathbf{X}$ when right-multiplied onto $\mathbf{X}$ (the first column of zeros and ones picks the second column of $\mathbf{X}$, and so on). The last column is $\mathbf{A}\overset{(n-1)}{\mathbf{x}}$ which is $\overset{(n)}{\mathbf{x}}$. However, $\overset{(n)}{\mathbf{x}}$ must be linearly dependent on the column vectors of $\mathbf{X}$, because $\mathbf{X}$ has already n linearly independent columns. The characteristic equation provides $\overset{(n)}{\mathbf{x}}$ via the Cayley-Hamilton theorem,

$$\overset{(n)}{\mathbf{x}} + a_{n-1}\overset{(n-1)}{\mathbf{x}} + \ldots + a_1\dot{\mathbf{x}} + a_0\mathbf{x} = \mathbf{0}$$

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Thus the rightmost column is completed as follows,

$$\mathbf{AX} = \mathbf{X}\begin{bmatrix} 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}$$

Since $\mathbf{X}$ is assumed to be full rank, it is invertible, and $\mathbf{X}^{-1}\mathbf{AX} = \mathbf{A}_o$, the square matrix in brackets. It is the $\mathbf{A}$ matrix in the companion form associated with the observable canonical form of a state space model. For convenience, it is here referred to as observable canonical form, although the current state space model has neither inputs nor outputs.

The relation $\mathbf{X}^{-1}\mathbf{AX} = \mathbf{A}_o$ ($\mathbf{X}$ full rank) transforms any $\mathbf{A}$ matrix to its observable canonical form. It has two benefits: (1) it reveals the coefficients of the characteristic polynomial, and (2) the system matrix is transformed to a simple, standard form. Numerical values may produce singular cases in practice, where some of the column vectors of $\mathbf{X}$ are linearly dependent, but the symbolic computations assume independence. The transformation maps similar system matrices (sharing the same characteristic polynomial) into the same form, which means the map is many-to-one.

To summarize: Even though the kinematic matrix can be generated from an arbitrary starting vector $\mathbf{x} = \mathbf{x}^*$, the resulting $\mathbf{A}_o$ is the same (the characteristic polynomial is invariant).

III. DEGREE REDUCTION

Now decompose the characteristic polynomial $p(D)$ into a *leading* part $q(D) = D^n + a_{n-2}D^{n-2} + a_{n-4}D^{n-4} + \ldots$ and a *lagging* part $r(D) = a_{n-1}D^{n-1} + a_{n-3}D^{n-3} + \ldots$, such that $p = q + r$. The degree of the leading part is one higher than the degree of the lagging part.

For exponential stability it is necessary that the ratio of the leading coefficients of q and r is positive. Forming the rational function $\frac{q}{r}$, the quotient is $\frac{1}{a_{n-1}}D$. Therefore $a_{n-1} > 0$ is required for stability, but that was assumed already. To complete the division, there is a remainder polynomial, which is also divided by r. This rational function must, in turn, also be positive with respect to the leading coefficients. The procedure continues recursively to the bottom, where the remainder is a constant. The degree-reduction algorithm applies long division, and it is equivalent to a continued-fraction expansion.

Conveniently, the degree-reduction algorithm lists the whole polynomial while working on the leading and lagging parts. Thus the state vector spans every step, which makes the algorithm suited for matrix-oriented programming.

It is a recursive algorithm, which reduces the polynomial degree by one at each step. The polynomial at each step must have positive coefficients for exponential stability. The algorithm stops when it reaches a second degree polynomial, which must have positive coefficients. The polynomial coefficients are precisely the elements of Routh's array.

The algorithm builds a sequence of polynomials. The new polynomial is the remainder $p_{i+1} = p_i - quotient \cdot lagpolynomial$. Polynomial p_i is exponentially stable if p_{i+1} is, therefore the coefficients of p_{i+1} must be positive.

The initial polynomial is $p_0 = p$. The first quotient $k_1 = \frac{1}{a_{n-1}}D$ results from p_0/r_0 where r_0 is the lagging polynomial. The numerator polynomial is one degree higher than the denominator, so the quotient is of the first degree. The coefficient is positive. The remainder $p_1 = p_0 - k_1 r_0$ is a new polynomial, which must have positive coefficients. Even the remainder consists of a leading and a lagging part, which must be tested, and the recursion continues. Hinrichsen and Pritchard (2005) [4] describe the algorithm rigorously (also Chapellat, Mansour and Bhattacharyya 1990 [1]).

A. Fourth Order Example

For example, the general fourth degree polynomial is $p = D^4 + a_3D^3 + a_2D^2 + a_1D + a_0$. Let $p_0 = p$. Again, assume all coefficients are positive. The leading polynomial is $q_0 = D^4 + a_2D^2 + a_0$ and the lagging polynomial is $r_0 = a_3D^3 + a_1D$.

The quotient from the division $\frac{p_0}{r_0}$ is $k_1 = \frac{1}{a_3}D$. The remainder is

$$\begin{aligned} p_1 &= p_0 - k_1 r_0 \\ &= D^4 + a_3D^3 + a_2D^2 + a_1D + a_0 \\ &\quad - \frac{1}{a_3}D\left(a_3D^3 + a_1D\right) \\ &= a_3D^3 + \left(a_2 - \frac{a_1}{a_3}\right)D^2 + a_1D + a_0 \end{aligned}$$

The remainder must have positive coefficients. Its leading polynomial is $q_1 = a_3D^3 + a_1D$ and its lagging polynomial is $r_1 = \left(a_2 - \frac{a_1}{a_3}\right)D^2 + a_0$. The quotient from the division $\frac{p_1}{r_1}$ is $k_2 = \frac{a_3}{a_2 - \frac{a_1}{a_3}}D$. The remainder is

$$\begin{aligned} p_2 &= p_1 - k_2 r_1 \\ &= a_3D^3 + \left(a_2 - \frac{a_1}{a_3}\right)D^2 + a_1D + a_0 \\ &\quad - \frac{a_3}{a_2 - \frac{a_1}{a_3}}D\left(\left(a_2 - \frac{a_1}{a_3}\right)D^2 + a_0\right) \\ &= \left(a_2 - \frac{a_1}{a_3}\right)D^2 + \left(a_1 - \frac{a_3a_0}{a_2 - \frac{a_1}{a_3}}\right)D + a_0 \end{aligned}$$

The following table summarizes the coefficients of the p_i polynomials row by row.

D^4	D^3	D^2	D^1	D^0
1	a_3	a_2	a_1	a_0
	a_3	$a_2 - \frac{a_1}{a_3}$	a_1	a_0
		$a_2 - \frac{a_1}{a_3}$	$a_1 - \frac{a_3a_0}{a_2 - \frac{a_1}{a_3}}$	a_0
			$a_1 - \frac{a_3a_0}{a_2 - \frac{a_1}{a_3}}$	a_0
				a_0

The coefficients in the diagonal must be positive. Consequently, *all* coefficients are positive (start from the bottom right corner and move up row-by-row).

IV. ALGORITHM *ROUTHMAT*

Assume the system matrix is in observable canonical form $\mathbf{A} = \mathbf{A}_o$, where

$$\mathbf{A}_o = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}$$

The algorithm is symbolic, and it is based on degree reduction and elimination.

Assume there is a transformation $\mathbf{R}_u^{-1}\mathbf{A}_o\mathbf{R}_u = \mathbf{A}_{Lc}$ that results in the following tridiagonal form,

$$\mathbf{A}_{Lc} = \begin{bmatrix} 0 & -f_{n-1} & & & \\ f_n & 0 & \ddots & & \\ & f_{n-1} & \ddots & -f_2 & \\ & & \ddots & 0 & -f_1 \\ & & & f_2 & -f_1 \end{bmatrix} \quad (1)$$

It is here referred to as a *column ladder canonical form*. Its flow graph has a ladder structure of touching cycles. Each column, except the first, contains two occurrences of the same factor f_i $(i = 1, 2, \dots, n-1)$.

Consider the relationship $\mathbf{A}_o\mathbf{R}_u = \mathbf{R}_u\mathbf{A}_{Lc}$ between the vectors $\mathbf{r}_i$ $(\mathbf{R}_u = [\mathbf{r}_n, \dots \mathbf{r}_2, \mathbf{r}_1])$, hereafter *Routh vectors*,

$$\mathbf{A}_o\mathbf{r}_{k+1} = -f_{k+2}\mathbf{r}_{k+2} + f_{k+2}\mathbf{r}_k \quad (k = 1, 2, \dots, n-2)$$

To explain, the velocity vector for a point at the position $\mathbf{r}_{k+1}$ is in the plane spanned by the preceding $\mathbf{r}_k$ and the succeeding $\mathbf{r}_{k+2}$. The coordinates are f_{k+2} and $-f_{k+2}$ as specified by the columns of the $\mathbf{A}_{Lc}$ matrix. The next Routh vector to be found, recursively, is the one with the higher index $\mathbf{r}_{k+2}$. Therefore rearrange to obtain the following recursion,

$$\mathbf{r}_{k+2} = \mathbf{r}_k - \mathbf{A}_o\mathbf{r}_{k+1}/f_{k+2} \quad (2)$$

The velocity $\mathbf{A}_o\mathbf{r}_{k+1}$ is scaled by the factor $1/f_{k+2}$, which is chosen to induce zeros below the diagonal of $\mathbf{R}_u$ (upper triangular, hence the subscript u). That is, the lowest nonzero element is eliminated, and $\mathbf{r}_{k+2}$ uses one coordinate less than $\mathbf{r}_{k+1}$. The final vector $\mathbf{r}_n$ will be parallel with the first coordinate axis.

The algorithm must use *two* initial vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ to get started. The first vector is the absolute value of the last column of $\mathbf{A}_o$, with every second element replaced by zero in the following manner,

$$\mathbf{r}_1 = \begin{bmatrix} \vdots \\ 0 \\ a_{n-3} \\ 0 \\ a_{n-1} \end{bmatrix} \quad (3)$$

According to $\mathbf{A}_{Lc}$, the second vector depends on $\mathbf{r}_1$ as follows,

$$\mathbf{r}_2 = -\mathbf{r}_1 - \mathbf{A}_o\mathbf{r}_1/f_1 \quad (4)$$

The factor $1/f_1$ is chosen to eliminate the lowest element $-a_{n-1}$ in $-\mathbf{r}_1$. Now the recursion can continue on the column vectors of $\mathbf{R}_u$ taken from right to left, using the machinery in (2). The last constant is special, because $\mathbf{A}_o\mathbf{r}_n = f_n\mathbf{r}_{n-1}$ according to $\mathbf{A}_{Lc}$. The vectors left and right of the equality sign contain only one element in the same position. The constant f_n is chosen to satisfy the proportionality.

The Routh vectors are stored in the following upper-triangular matrix,

$$\mathbf{R}_u = \begin{bmatrix} r_{1,1} & & & & \\ & \ddots & \vdots & \vdots & \vdots \\ & & 0 & r_{n-3,n-1} & 0 \\ & & r_{n-2,n-2} & 0 & r_{n-2,n} \\ & & & r_{n-1,n-1} & 0 \\ & & & & r_{n,n} \end{bmatrix} \quad (5)$$

The nonzero elements above the diagonal are in a chequerboard pattern of zeroes. They are the Routh-array elements, although Routh's array is more compact and not a square matrix. Still, $\mathbf{R}_u$ is referred to as the *Routh matrix* in what follows, being a matrix version of Routh's array. According to Routh's stability criterion, all diagonal elements of $\mathbf{R}_u$ must be positive for the system to be exponentially stable – and conversely. In fact, *all* the nonzero elements are positive if and only if the system is exponentially stable owing to the degree reduction process. Any two neighbouring column vectors are orthogonal $\left(\mathbf{r}_{k+1}^T\mathbf{r}_k = 0\right)$. Odd indexed columns are orthogonal to all the even indexed columns and vice versa.

The following transformation brings out $\mathbf{A}_{Lc}$ by construction,

$$\mathbf{R}_u^{-1}\mathbf{A}_o\mathbf{R}_u = \mathbf{A}_{Lc} \quad (6)$$

Inversely, the inverse Routh matrix transforms $\mathbf{A}_{Lc}$ to observable canonical form. In fact, $\mathbf{R}_u^{-1}$ is a kinematic matrix for the ladder form $\mathbf{A}_{Lc}$.

Algorithm 1 starts arbitrarily from the observable canonical form. The transpose of the equation is valid for the system matrix on controllable canonical form $\mathbf{A}_c = \mathbf{A}_o^T$. That is,

$$\mathbf{R}_u^T\mathbf{A}_c\left(\mathbf{R}_u^{-1}\right)^T = \mathbf{A}_{Lc}^T \triangleq \mathbf{A}_{Lr} \quad (7)$$

Algorithm 1 Algorithm RouthMat. It derives the Routh vectors from the observable canonical form.

1) Transform the matrix to observable canonical form $\mathbf{A}_o$ (Section II).
2) Compute the first Routh vector $\mathbf{r}_1$ as in (3). The first factor is $f_1 = a_{n-1}$.
3) Find the second factor f_2 as described; but if the order $n = 2$ then $f_2 = \frac{a_0}{a_1}$, and if $n > 2$ then $f_2 = \frac{a_{n-2} - \frac{a_{n-3}}{a_{n-1}}}{a_{n-1}}$.
4) Compute the second Routh vector $\mathbf{r}_2$ as in (4).
5) Compute the remaining Routh vectors by the recursion formula in (2). Compute first the vector $\mathbf{A}_o\mathbf{r}_{k+1}$. Then select f_{k+2} to eliminate the nonzero coordinate below the diagonal.
6) Compute the last factor f_n such that $\mathbf{A}_o\mathbf{r}_n = f_n\mathbf{r}_{n-1}$ is satisfied.
7) Store all Routh vectors in $\mathbf{R}_u = [\mathbf{r}_n, \ldots \mathbf{r}_2, \mathbf{r}_1]$, filling it from right to left as in (5).
8) Store the factors in the diagonal matrix $\mathbf{D} = \mathrm{Diag}\left[\frac{1}{f_n}, \frac{1}{f_{n-1}}, \ldots, \frac{1}{f_1}\right]$.
9) Perform the transformation in (6) to check the structure of the result.

The matrix $\mathbf{A}_{Lr}$ is the corresponding *row* ladder canonical form. Similarly, the transpose $\mathbf{R}_u^T \triangleq \mathbf{R}_\ell$, which is lower triangular. The Routh matrix $\mathbf{R}_\ell$ could have been derived starting from $\mathbf{A}_c$ instead of starting from the observable canonical form.

A. Fourth Order Example

The general characteristic polynomial is $p = D^4 + a_3D^3 + a_2D^2 + a_1D + a_0$. The system matrix is in observable form (Algorithm 1, Step 1),

$$\mathbf{A}_o = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}$$

The first Routh vector (Step 2) depends on the last column of the matrix, corresponding to the lagging part of the characteristic polynomial,

$$\mathbf{r}_1 = \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_3 \end{bmatrix}$$

The first factor is $f_1 = a_{n-1} = a_3$. The second factor (Step 3) is $f_2 = \frac{a_{n-2} - \frac{a_{n-3}}{a_{n-1}}}{a_{n-1}} = \frac{a_2 - \frac{a_1}{a_3}}{a_3}$ $(n = 4)$.

From (4) the second Routh vector (Step 4) is the following,

$$\mathbf{r}_2 = \begin{bmatrix} a_0 \\ 0 \\ \frac{a_2a_3 - a_1}{a_3} \\ 0 \end{bmatrix}$$

The recursion continues (Steps 5 and 6), and all vectors are stored in the Routh matrix (Step 7),

$$\mathbf{R}_u = \begin{bmatrix} a_0 & 0 & a_0 & 0 \\ 0 & -\frac{a_0a_3^2}{a_2a_3 - a_1} + a_1 & 0 & a_1 \\ 0 & 0 & \frac{a_2a_3 - a_1}{a_3} & 0 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

The factors f_i^{-1} are stored in the diagonal matrix $\mathbf{D}$ (Step 8), and the factors f_i appear in the system matrix $\mathbf{A}_{Lc}$ (Step 9), see Fig. 1.

The Routh matrix $\mathbf{R}_u$ is upper triangular as expected. All elements are positive if and only if the system is exponentially stable.

V. Electrical Analogue

If all the characteristic roots have negative real parts, an electrical RLC ladder network possessing that characteristic polynomial can be realized. Figure 2 shows a ladder generated by means of Cauer synthesis [5]. The synthesis relies on the same continued fraction that also can be used to test stability [3]. Owing to this common feature, the matrix $\mathbf{D}$ contains directly the component values of the ladder network.

In general terms the voltage drop over an inductor is $v = L\frac{\mathrm{d}i}{\mathrm{d}t} = LDi$, where D is the differential operator. The voltage drop over a capacitor is $v = \frac{1}{C}\int i = \frac{1}{C}\frac{1}{D}i$. In accordance with the figure, v_1 is the voltage drop across the capacitor C_1, i_2 is the current through the inductor L_2 (left-to-right), v_3 is the voltage drop across the capacitor C_3, and i_4 is the current through the inductor L_4 (left-to-right). The component relationships give the following set of equations,

$$\begin{aligned} v_1 = -\frac{i_2}{DC_1} \Rightarrow Dv_1 = -\frac{i_2}{C_1} \quad &\Rightarrow \quad \dot{v}_1 = -\frac{1}{C_1}i_2 \\ L_2Di_2 = v_1 - v_3 \quad &\Rightarrow \quad \dot{i}_2 = \frac{1}{L_2}v_1 - \frac{1}{L_2}v_3 \\ v_3 = \frac{1}{DC_3}(i_2 - i_4) \quad &\Rightarrow \quad \dot{v}_3 = \frac{1}{C_3}i_2 - \frac{1}{C_3}i_4 \\ L_4Di_4 = v_3 - Ri_4 \quad &\Rightarrow \quad \dot{i}_4 = \frac{1}{L_4}v_3 - \frac{R}{L_4}i_4 \end{aligned}$$

Choosing the state vector $\mathbf{y} = [v_1\ i_2\ v_3\ i_4]^T$ of electrical variables, an autonomous state-space model appears in the compact form $\dot{\mathbf{y}} = \mathbf{A}_y\mathbf{y}$, where

$$\mathbf{A}_y = \begin{bmatrix} 0 & -\frac{1}{C_1} & 0 & 0 \\ \frac{1}{L_2} & 0 & -\frac{1}{L_2} & 0 \\ 0 & \frac{1}{C_3} & 0 & -\frac{1}{C_3} \\ 0 & 0 & \frac{1}{L_4} & -\frac{R}{L_4} \end{bmatrix}$$

The system matrix is in a row ladder form as a consequence of Ohm's and Kirchhoff's laws.

The Routh matrix connects the row ladder form and the controllable canonical form in the following manner,

$$\mathbf{A}_{Lr} = \left(\mathbf{R}_u^{-1}\mathbf{A}_o\mathbf{R}_u\right)^T = \mathbf{R}_\ell\mathbf{A}_c\mathbf{R}_\ell^{-1}, \quad \mathbf{R}_\ell = \mathbf{R}_u^T$$

The Routh matrix depends only on $\mathbf{A}_o = \mathbf{A}_c^T$ ($\mathbf{R}_u$ and $\mathbf{D}$ are derived earlier). Furthermore,

$$\mathbf{D}=\begin{bmatrix} \frac{a_1a_2a_3-a_0a_3^2-a_1^2}{(a_2a_3-a_1)a_0} & 0 & 0 & 0 \\ 0 & \frac{(a_2a_3-a_1)^2}{\left(a_1a_2a_3-a_0a_3^2-a_1^2\right)a_3} & 0 & 0 \\ 0 & 0 & \frac{a_3^2}{a_2a_3-a_1} & 0 \\ 0 & 0 & 0 & \frac{1}{a_3} \end{bmatrix}$$

$$\mathbf{A}_{Lc}=\begin{bmatrix} 0 & -\frac{\left(a_1a_2a_3-a_0a_3^2-a_1^2\right)a_3}{(a_2a_3-a_1)^2} & 0 & 0 \\ \frac{(a_2a_3-a_1)a_0}{a_1a_2a_3-a_0a_3^2-a_1^2} & 0 & -\frac{a_2a_3-a_1}{a_3^2} & 0 \\ 0 & \frac{\left(a_1a_2a_3-a_0a_3^2-a_1^2\right)a_3}{(a_2a_3-a_1)^2} & 0 & -a_3 \\ 0 & 0 & \frac{a_2a_3-a_1}{a_3^2} & -a_3 \end{bmatrix}$$

$$Z=\frac{numerator}{D^4+\frac{R}{L_4}D^3+\left(\frac{1}{C_3L_4}+\frac{1}{L_2C_3}+\frac{1}{C_1L_2}\right)D^2+\left(\frac{R}{L_2C_3L_4}+\frac{R}{C_1L_2L_4}\right)D+\frac{1}{C_1L_2C_3L_4}}$$

Fig. 1. Wide equations from the fourth order example (Sec. IV-A) and its electrical analogue (Sec. V).

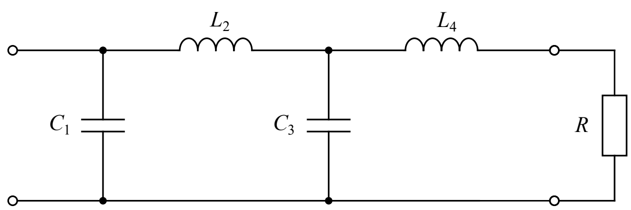

Fig. 2. A fourth-order lowpass filter. The ladder network passes fast frequencies through the shunt capacitors. Slow frequencies pass through the series inductors.

$$\mathbf{D} = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}$$

To verify, the input impedance of the ladder (from left to right) is Z, see Fig. 1. The denominator is the characteristic polynomial. Inserting the values from $\mathbf{D}$ (Fig. 1) confirms that the coefficients of the denominator match the original a_i $(i = 0, 1, 2, 3)$. The $\mathbf{D}$ matrix assumes the resistor $R = 1$.

If $R \neq 1$, then L_4 must be multiplied by R, the following component C_3 must be divided by R, the following L_2 multiplied by R, and the following C_1 divided by R. The resistor value propagates in an alternating manner owing to the underlying polynomial divisions. In short, inductances are multiplied by R and capacitances are divided by R.

The matrices $\mathbf{R}_u$ and $\mathbf{D}$ come out as a result of the Routh matrix algorithm. Alternatively, they can be computed by hand using the degree-reduction algorithm or the continued-fraction expansion.

The network is passive, and the resistor dissipates energy. Therefore the ladder is exponentially stable. If $R = 0$, the ladder consists of reactive components only, and it will not be exponentially stable; but it could be marginally stable.

Routh's stability criterion is equivalent to the ladder network having positive components.

VI. DISCUSSION

For the fourth degree polynomial, the following is a sketch of the standard Routh array:

$$\begin{array}{c|ccc} D^4 & 1 & a_2 & a_0 \\ D^3 & a_3 & a_1 & 0 \\ D^2 & b_1 & b_2 & 0 \\ D^1 & c_1 & 0 & 0 \\ D^0 & d_1 & 0 & 0 \end{array}$$

The three last rows are derived from the first two. The Routh matrix, on the other hand, reorganizes the elements into the following square matrix:

$$\mathbf{R}_u = \begin{bmatrix} d_1 & 0 & b_2 & 0 \\ 0 & c_1 & 0 & a_1 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & a_3 \end{bmatrix}$$

Omitting the top row of Routh's array, each column of $\mathbf{R}_u$ corresponds to an array row padded with zeros. The Hurwitz test matrix, of which the principal minor determinants govern stability, is a square matrix with the last n polynomial coefficients along the diagonal. However, its axes are not compatible with the state vector. The new Routh matrix is, and it defines a change of basis, where the column vectors can be seen as axes.

The continued fraction is a shared feature of the stability criterion and the Cauer synthesis of the ladder network; this is a unique coupling. The Cauer synthesis of the ladder network starts with the admittance $\frac{1}{Z} = \frac{q}{r}$, where q is one degree higher than r. The continued-fraction expansion of the impedance has the following form:

$$Z = \frac{1}{C_1D + \frac{1}{L_2D + \frac{1}{C_3D + \frac{1}{L_4D}}}}$$

The expansion ends with a zero, and the resulting network ends with a short circuit $(R = 0)$. That network is marginally stable (p is assumed exponentially stable), and its characteristic polynomial is q. It is only when $R \neq 0$ that the characteristic polynomial becomes $p = q + r$ as evidenced by the denominator of Z in Fig. 1. An alternative network starts with a series inductor and ends with a capacitor, which is then loaded with R (in parallel across the capacitor). It is the result of expanding $Z = \frac{q}{r}$ instead. The lower right corner element of the $\mathbf{A}_y$ matrix would then have the form $-\frac{1}{RC}$ instead of $-\frac{R}{L}$.

The derivations are elementary in the sense that they rely on fundamental principles. Hence the list of references is relatively short. Readers, or students, with a standard background should be able to replicate the steps. The novelty is in the new, active role of Routh's criterion, and in the connection the Routh matrix makes to RLC ladder networks.

The matrix $\mathbf{A}_y$ does not have quite the form of $\mathbf{A}_{Lr}$; the nondiagonal elements have opposite signs. A match is achieved by a further coordinate transformation,

$$\mathbf{S} = \mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

such that $\mathbf{S}\mathbf{R}_\ell \mathbf{A}_c \mathbf{R}_\ell^{-1} \mathbf{S}^{-1} = \mathbf{A}_y$. Formally, it is achieved by assigning a counterclockwise sense to the currents of the two meshes, such that $\mathbf{A}_y$ changes its signs. The choice of sense of the reference currents is arbitrary in network theory.

The network is a passive, dissipative circuit. Let energy coordinates $\mathbf{z} = \mathbf{D}^{\frac{1}{2}}\mathbf{y}$. Then the sum of the stored electrical and magnetic energy is $E = \frac{1}{2}\mathbf{z}^T\mathbf{z}$. This is a natural Lyapunov function. Its derivative is the dissipation in the resistor, or $\dot{E} = -Ri_4^2$. Inverse coordinate transformations can take the Lyapunov function back to state-space coordinates; or it can be taken to any other basis, if need be. The electrical coordinates are $\mathbf{y} = \mathbf{S}\mathbf{R}_\ell\mathbf{x}$, where $\mathbf{x}$ are the phase variables. The Lyapunov function is by substitution $\frac{1}{2}\mathbf{y}^T\mathbf{D}\mathbf{y} = \frac{1}{2}\mathbf{x}^T\mathbf{R}_\ell^T\mathbf{S}^T\mathbf{D}\mathbf{S}\mathbf{R}_\ell\mathbf{x}$. But the sign changes cancel out on the diagonal $\mathbf{D}$ (the coordinates are squared), and the Lyapunov function in phase variables is just $\frac{1}{2}\mathbf{x}^T\mathbf{R}_\ell^T\mathbf{D}\mathbf{R}_\ell\mathbf{x}$. This is a general result pertaining to the controllable canonical form.

The ladder network is a lowpass filter, and filter theory can be exploited and transferred to state space. For pole placement, filter theory provides several established eigenvalue configurations with well-known properties in the frequency domain. For example, a filter can be designed with a prescribed cutoff frequency (Butterworth filter), such that a step response can be achieved according to specifications. Extending the ladder with more stages (sections consisting of an L and a C) achieves a sharper rejection of unwanted frequencies. Oppositely, for model order reduction, removing one or more stages may result in a smaller model with acceptable performance.

The stability criterion is equivalent to having all RLC components positive. How to accommodate marginally stable and unstable models is yet to be investigated; for simplicity, the scope of the current work is exponentially stable models. The numerical condition that results in a zero in the first column of the Routh array has not been analyzed. Its consequences for the analogy must be clarified.

The performance of the algorithm (e.g., computing time, memory consumption, and convergence) has not yet been addressed, as numerical analysis was omitted from this study. Particularly, the Krylov sequence, which can be seen as a preliminary stage to the core algorithm, is prone to ill-conditioning.

For now, the emphasis is on the new algebraic relationships enabled by the Routh matrix. An advantage is that sensitivities can be calculated algebraically, in contrast to working with eigenvalues. Robustness and energy flow can be evaluated with respect to the local resonance frequencies of each LC stage.

VII. CONCLUSIONS

The Routh matrix algorithm adds a state-space-compatible matrix to the inventory of established stability criteria. It is a mere reorganization of the standard Routh array, and it comes out as a result of the relationship $\mathbf{A}_o\mathbf{R}_u = \mathbf{R}_u\mathbf{A}_{Lc}$, where $\mathbf{A}_{Lc}$ is crafted in a particular manner. Its immediate relevance is to define a change of basis. The new basis leads to an electrical analogue, which will be relevant for a design approach based on electrical network theory.

Due to the fundamental nature of Routh's stability criteria, there are several possible application areas. For example: Lyapunov functions, pole placement, frequency-response shaping, robust control, model-order reduction (and expansion), and state-space control education.

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REFERENCES

[1] H. Chapellat, M. Mansour, and S. P. Bhattacharyya, Elementary proofs of some classical stability criteria, IEEE Trans. Education, vol. 33, no. 3, pp. 232-239, August 1990.

[2] C. F. Chen, A new formulation of the Hermite criterion, Int. J. of Control, vol. 19, no. 4, pp. 757-764, 1974.

[3] J. J. DiStefano, A. R. Stubberud, and I. J. Williams, Feedback And Control Systems (2nd ed.), Schaum's Outline Series. New York: McGraw-Hill, 1990.

[4] D. Hinrichsen and A. J. Pritchard, Mathematical Systems Theory I. Berlin: Springer, 2005.

[5] M. E. Van Valkenburg, Introduction to Modern Network Synthesis. New York, London: Wiley, 1960.

[6] H. S. Wall, Polynomials whose zeros have negative real parts, The American Mathematical Monthly, vol. 52, no. 6, pp. 308-322, 1945.

[7] R. Yarlagadda, An application of tridiagonal matrices to network synthesis, SIAM J. Appl. Math, vol. 16, no. 6, pp. 1146-1162, November 1968.