Extending Fixed-Point Theory for Neural Field Models

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***Abstract*—This paper studies operators in neural field models for which the classical Banach contraction principle does not apply. We show that Kannan's fixed-point condition can still guarantee the existence and uniqueness of fixed points, even when the operator is discontinuous or fails to be globally Lipschitz. To illustrate this, we first consider a simple scalar operator on a bounded interval for which Banach's principle fails, but the Kannan inequality holds. We then analyze a smooth neural field operator and a neural field with a Heaviside activation, deriving Kannan-type estimates on relevant subsets. These examples show how Kannan-type estimates may provide an alternative framework for studying fixed points in neural field models that fall outside the scope of Banach's classical theory.**

***Index Terms*—Fixed-point theory, Kannan-type contraction, Neural field models, Heaviside activation.**

I. Introduction

The mathematical modeling of neural activity has a rich history spanning nearly a century. Neural Field Theory has emerged as a fundamental framework for describing the large-scale dynamics of cortical tissue. Early continuum approaches to neural tissue were pioneered by Beurle [1], while Amari's groundbreaking work [2] modeled homogeneous neural fields with lateral inhibition, deriving their dynamics and classifying excitation patterns (monostable, bistable, periodic) based on connectivity and stimulus. He later extended the analysis to two-layer fields, revealing mechanisms for oscillations and traveling waves.

Subsequent developments by Wilson and Cowan [3], [4] extended this framework by modeling excitatory-inhibitory population dynamics. Nunez [5], [6] further applied neural field theory to understand electroencephalogram (EEG) rhythms and traveling waves. From a dynamical systems perspective, Ermentrout [7] applied bifurcation and perturbation methods to analyze pattern formation in spatially organized neural fields. In later comprehensive work, Ermentrout and Terman [8] synthesized methods from nonlinear dynamics including numerical, analytical, and multi-scale techniques to model and explain complex neuronal activity patterns, forming a cornerstone of modern mathematical neuroscience.

In recent decades, theoretical understanding and applications of neural field models have advanced considerably. The paper [9] provided a rigorous mathematical treatment of continuum neural field dynamics, including pattern formation, waves, and stochastic effects. Building on this foundation, the work [10] broadened the framework by integrating computational methods and biologically motivated applications across sensory, cognitive, and clinical domains. Rigorous functional analytic foundations for neural field operators were established by [11], [12], paving the way for advanced mathematical analysis.

Fixed point theory plays a central role in the analysis of neural field equations. The classical Banach Contraction Principle ensures existence and uniqueness for an operator $T : X \to X$ on a complete metric space (X, d) if there exists $\gamma \in (0, 1)$ such that

$$d(Tx, Ty) \leq \gamma\, d(x, y), \quad \forall x, y \in X. \tag{1}$$

While effective for smooth neural interactions and firing rate functions [13], the Banach Contraction Principle enforces a global Lipschitz condition with constant strictly less than one, which excludes neural field equations exhibiting stiff nonlinearities, high-contrast external inputs, or parameter-dependent bifurcations, and therefore fails to capture the multiplicity of stationary solutions observed in biologically realistic models.

Kannan's alternative contraction principle [14] provides a different fixed-point framework. An operator $T : X \to X$ satisfies the Kannan contraction property if there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma \left(d(x, Tx) + d(y, Ty) \right), \quad \forall x, y \in X. \tag{2}$$

In this article, we focus on simple classes of neural field operators for which Banach's contraction principle does not apply, but Kannan-type estimates can still be verified on natural subsets.

The rest of the paper is organized as follows. Section II presents a simple scalar operator on [0,1], illustrating that Banach's contraction principle fails while a Kannan-type inequality still holds. Section III analyzes a smooth neural field model, establishes that the operator is not a Banach contraction, and derives a Kannan-type estimate for constant profiles. Section IV extends the analysis to a neural field with a Heaviside firing rate, reduces the operator to a two-block structure, verifies the Kannan inequality, and proves the existence and uniqueness of a fixed point. Section V concludes by summarizing the results and their implications for neural field analysis.

II. Motivating Example

To illustrate the relevance of Kannan contractions, we begin with a very simple operator that fails to satisfy Banach's contraction principle, yet fulfills a Kannan-type inequality.

Let (E, d) the metric space be chosen as

$$E = [0,1] \subset \mathbb{R}, \quad d(x,y) = |x-y|.$$

Define the operator $T : E \to E$ by

$$Tx = \begin{cases} \dfrac{x}{4}, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \dfrac{x}{5}, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

a) Failure of Banach contraction: First observe that T is discontinuous at $x = \frac{1}{2}$ since

$$\lim_{x \to (\frac{1}{2})^-} Tx = \frac{1}{8}, \quad T\big(\tfrac{1}{2}\big) = \frac{1}{10}.$$

Hence T is not continuous. Hence T is not a Banach contraction on E.

b) Verification of the Kannan condition: We now verify that T satisfies the Kannan inequality

$$|Tx - Ty| \le \gamma\big(|x - Tx| + |y - Ty|\big), \quad \forall x, y \in E,$$

for a suitable constant $\gamma \in [0, \frac{1}{2})$. Indeed, for $x, y \in \left[0, \frac{1}{2}\right)$, we have

$$|Tx - Ty| = \frac{1}{4}|x - y|.$$

Moreover,

$$|x - Tx| + |y - Ty| = \frac{3}{4}x + \frac{3}{4}y.$$

Hence,

$$|Tx - Ty| \le \frac{1}{4}(x + y) \le \frac{4}{9}\big(|x - Tx| + |y - Ty|\big).$$

For $x, y \in \left[\frac{1}{2}, 1\right]$, it holds

$$|Tx - Ty| = \frac{1}{5}|x - y|.$$

Hence,

$$|x - Tx| + |y - Ty| = \frac{4}{5}x + \frac{4}{5}y,$$

and therefore

$$|Tx - Ty| \le \frac{1}{5}(x + y) \le \frac{4}{9}\big(|x - Tx| + |y - Ty|\big).$$

It remains to consider the mixed case. Let $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$. Then

$$|Tx - Ty| = \left|\frac{x}{4} - \frac{y}{5}\right| \le \frac{x}{4} + \frac{y}{5}.$$

On the other hand,

$$|x - Tx| + |y - Ty| = \frac{3}{4}x + \frac{4}{5}y.$$

Thus,

$$|Tx - Ty| \le \frac{4}{9}\big(|x - Tx| + |y - Ty|\big).$$

The case $y \in [0, \frac{1}{2})$ and $x \in [\frac{1}{2}, 1]$ is symmetric. Finally,

$$|Tx - Ty| \le \frac{4}{9}\big(|x - Tx| + |y - Ty|\big), \quad x, y \in E.$$

Since $\frac{4}{9} < \frac{1}{2}$, the operator T is a Kannan contraction.

III. Neural Field Model as a Kannan-Type Operator

We now examine a smooth neural field model that exhibits Kannan-type behavior on a relevant subset, while failing to be a Banach contraction in the standard supremum norm. This model is motivated by continuous attractor neural networks [2], [9].

A. Model Setup

a) Model: Let $\Omega = [0,1]$ and $X = C(\Omega)$ with the supremum norm

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$

Consider a neural–type operator

$$(Tu)(x) = I(x) + \int_0^1 w(x,y)\,\phi(u(y))\,dy, \quad x \in [0,1],$$

where:

- $I \in C(\Omega)$ is a given external input;
- $w \colon \Omega \times \Omega \to \mathbb{R}$ is a smooth synaptic kernel with

$$w(x,y) \ge 0, \quad \int_0^1 w(x,y)\,dy = 1 \quad \text{for all } x \in \Omega;$$

- $\phi \colon \mathbb{R} \to \mathbb{R}$ is a smooth activation function, defined as follows.

Fix constants $a > 0$ and $0 < \delta < \frac{a}{4}$, and let $I_0 \in \mathbb{R}$ be such that

$$I(x) \equiv I_0, \quad |a - I_0| > 2\delta,$$

while $\rho \in C^\infty(\mathbb{R}; [0,1])$ satisfies

$$\rho(u) = 1 \text{ for } |u| \le \delta, \quad \rho(u) = 0 \text{ for } |u| \ge 2\delta.$$

Define the activation by

$$\phi(u) = \rho(u)\,\big(u - a\big) + \big(1 - \rho(u)\big)\,\tanh(u).$$

Then $\phi \in C^\infty(\mathbb{R})$, and:

- for small potentials $|u| \le \delta$ we have

$$\phi(u) = u - a \quad \text{(linear regime with a strong offset)};$$

- for large $|u| \ge 2\delta$ it holds

$$\phi(u) = \tanh(u) \quad \text{(smooth saturation)};$$

- in the transition strip $\delta < |u| < 2\delta$ the function ϕ smoothly interpolates between these regimes.

b) 1. T is not a Banach contraction in $\|\cdot\|_\infty$: Take $I(x) \equiv I_0$ as above and consider constant profiles

$$u(x) \equiv c, \quad v(x) \equiv d$$

with $c, d \in [-\delta, \delta]$. Then for all $x \in \Omega$:

$$\phi(u(y)) = c - a, \quad \phi(v(y)) = d - a.$$

Using the normalization $\int_0^1 w(x,y)\,dy = 1$, we get

$$(Tu)(x) = I_0 + c - a, \quad (Tv)(x) = I_0 + d - a.$$

Hence Tu and Tv are constant functions, and

$$\|Tu - Tv\|_\infty = |(I_0 + c - a) - (I_0 + d - a)| = |c - d| = \|u - v\|_\infty.$$

Thus the global Lipschitz constant of T in $\|\cdot\|_\infty$ is at least 1, and no inequality of the form

$$\|Tu - Tv\|_\infty \le L\,\|u - v\|_\infty, \quad L < 1,$$

can hold for all $u, v \in X$. So T is not a Banach contraction on $(X, \|\cdot\|_\infty)$.

c) 2. Kannan-type structure on constant profiles: For a constant profile $u(x) \equiv c$ with $|c| \le \delta$ we have

$$Tu(x) = I_0 + c - a,$$

hence

$$u(x) - Tu(x) = c - (I_0 + c - a) = a - I_0,$$
$$\|u - Tu\|_\infty = |a - I_0|.$$

In particular, for such constant profiles the residual $\|u - Tu\|_\infty$ is independent of c and, by the assumption $|a - I_0| > 2\delta$, strictly larger than any possible state difference $|c - d| \le 2\delta$ in this regime.

Take again u, v as above with $c, d \in [-\delta, \delta]$. Then

$$\|Tu - Tv\|_\infty = |c - d| \le 2\delta <$$
$$|a - I_0| = \|u - Tu\|_\infty = \|v - Tv\|_\infty.$$

Therefore

$$\|Tu - Tv\|_\infty \le \frac{2\delta}{|a - I_0|}\,\|u - Tu\|_\infty.$$

Hence

$$\|Tu - Tv\|_\infty \le \frac{\delta}{|a - I_0|}\,(\|u - Tu\|_\infty + \|v - Tv\|_\infty).$$

Since $|a - I_0| > 2\delta$, we have

$$q = \frac{\delta}{|a - I_0|} < \frac{1}{2}.$$

Then, for all constant profiles u, v with values in $[-\delta, \delta]$, we obtain

$$\|Tu - Tv\|_\infty \le q\big(\|u - Tu\|_\infty + \|v - Tv\|_\infty\big).$$

Thus the operator T is *not* a Banach contraction in $\|\cdot\|_\infty$, but it satisfies a Kannan-type estimate on this class of constant profiles. This estimate should not be interpreted as a direct application of Kannan's fixed-point theorem, since the class of constant profiles with values in $[-\delta, \delta]$ is not invariant under T under the present assumptions.

B. A Posteriori Error Estimate for a Kannan Contraction

Assume that $D \subset C(\Omega)$ is a nonempty closed set, $T(D) \subset D$, and T satisfies the Kannan contraction condition on D with a constant $\sigma \in (0, \frac{1}{2})$. Let $u^* \in D$ denote the unique fixed point of T in D. Then, for any $u \in D$, the following residual estimate holds:

$$\|u^* - u\|_\infty \le \frac{1 - \sigma}{1 - 2\sigma}\|Tu - u\|_\infty.$$

Indeed, set $u_0 = u$ and $u_{k+1} = Tu_k$. By the Kannan inequality,

$$\|u_{k+1} - u_k\|_\infty \le \sigma\big(\|u_k - u_{k+1}\|_\infty + \|u_{k-1} - u_k\|_\infty\big), \quad k \ge 1.$$

Hence

$$\|u_{k+1} - u_k\|_\infty \le \frac{\sigma}{1 - \sigma}\|u_k - u_{k-1}\|_\infty.$$

Since

$$r = \frac{\sigma}{1 - \sigma} < 1,$$

we obtain

$$\|u_{k+1} - u_k\|_\infty \le r^k\|Tu - u\|_\infty.$$

Therefore,

$$\|u^* - u\|_\infty \le \sum_{k=0}^{\infty}\|u_{k+1} - u_k\|_\infty \le \frac{1}{1 - r}\|Tu - u\|_\infty.$$

Since

$$\frac{1}{1 - r} = \frac{1 - \sigma}{1 - 2\sigma},$$

the required estimate follows.

IV. Neural Field Model with Heaviside Activation as a Kannan-Type Operator

We present an example of a neural field operator with a Heaviside firing rate that satisfies a Kannan-type contraction property but is not a Banach contraction. The example is constructed so that the integral operator maps every function into a two-block function that is constant on two subintervals of the domain. As a result, the problem reduces to a map on $\mathbb{R}^2$, which allows the Kannan inequality to be verified directly and the fixed point to be identified.

Let $\Omega = [0, 1]$ and set $X = L^\infty(\Omega)$ with the essential supremum norm. Partition Ω into $\Omega_1 = [0, \frac{1}{2}]$ and $\Omega_2 = (\frac{1}{2}, 1]$. Consider

$$(Tu)(x) = -\frac{1}{2}\int_0^1 w(x,y)\,H(u(y) - 2)\,dy,$$

where $H(s) = 0$ for $s \le 0$ and $H(s) = 1$ for $s > 0$.

Assume

$$w(x,y) = \begin{cases} 2, & (x,y) \in \Omega_1 \times \Omega_1 \text{ or } (x,y) \in \Omega_2 \times \Omega_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then w is bounded and measurable and $H(u(\cdot) - 2)$ is measurable and bounded for every $u \in L^\infty(\Omega)$, hence $T : L^\infty(\Omega) \to L^\infty(\Omega)$ is well defined. Moreover, for a.e. $x \in \Omega$,

$$|(Tu)(x)| \le \frac{1}{2}\int_0^1 |w(x,y)|\,dy = \frac{1}{2}, \quad \|Tu\|_\infty \le \frac{1}{2}.$$

For $(p,q)\in\mathbb{R}^2$ define the two-block class

$$X_2=\left\{u_{p,q}\in L^\infty(\Omega)\colon \begin{array}{l} u_{p,q}(x)=p \text{ a.e. on } \Omega_1,\\ u_{p,q}(x)=q \text{ a.e. on } \Omega_2,\end{array}\right\}.$$

With the norm $\|(p,q)\|_\infty=\max\{|p|,|q|\}$ on $\mathbb{R}^2$, the identification $J(u_{p,q})=(p,q)$ is an isometry, hence X_2 is closed and complete.

Lemma 1. *For every $u\in L^\infty(\Omega)$, one has $Tu\in X_2$. In particular, $T(X_2)\subset X_2$.*

Proof. If $x\in\Omega_1$, then $w(x,y)=2$ on Ω_1 and $w(x,y)=0$ on Ω_2, so for a.e. $x\in\Omega_1$,

$$(Tu)(x)=-\frac{1}{2}\int_{\Omega_1}2\,H(u(y)-2)\,dy,$$

which is independent of x. Thus Tu is constant a.e. on Ω_1. The same argument on Ω_2 gives that Tu is constant a.e. on Ω_2. Hence $Tu\in X_2$. □

For $u=u_{p,q}\in X_2$, we get

$$(Tu)(x)=-\frac{1}{2}H(p-2) \text{ a.e. on } \Omega_1,$$

$$(Tu)(x)=-\frac{1}{2}H(q-2) \text{ a.e. on } \Omega_2.$$

Thus $T|_{X_2}$ reduces to $F:\mathbb{R}^2\to\mathbb{R}^2$,

$$F(p,q)=(f(p),f(q)),\quad f(t)=-\frac{1}{2}H(t-2).$$

Proposition 1. *The map F is not a Banach contraction on $(\mathbb{R}^2,\|\cdot\|_\infty)$, but it satisfies*

$$\|F(\xi)-F(\eta)\|_\infty\le\frac{1}{5}\big(\|\xi-F(\xi)\|_\infty+\|\eta-F(\eta)\|_\infty\big)$$
$$\forall\xi,\eta\in\mathbb{R}^2.$$

Proof. Since f is discontinuous at 2, it cannot be a Banach contraction on $\mathbb{R}$. Consequently, $F=(f,f)$ cannot be a Banach contraction on $\mathbb{R}^2$.

For $a,b\in\mathbb{R}$, the Kannan inequality for f is trivial if $a,b\le 2$ or $a,b>2$. If $a\le 2<b$, then $|f(a)-f(b)|=\frac{1}{2}$ and

$$|a-f(a)|+|b-f(b)|=|a|+b+\tfrac{1}{2}>\tfrac{5}{2},$$

hence

$$|f(a)-f(b)|<\tfrac{1}{5}\big(|a-f(a)|+|b-f(b)|\big).$$

The case $b\le 2<a$ is symmetric. Thus

$$|f(a)-f(b)|\le\frac{1}{5}\big(|a-f(a)|+|b-f(b)|\big)\quad\forall a,b\in\mathbb{R}.$$

Applying this estimate to each coordinate and using the relation

$$\max_i(A_i+B_i)\le\max_i A_i+\max_i B_i$$

for nonnegative A_i,B_i, we obtain the stated inequality for F. □

Theorem 1. *The operator T has a unique fixed point in $L^\infty(\Omega)$.*

Proof. By the lemma, every fixed point of T belongs to X_2. Since X_2 is complete and $T|_{X_2}$ is isometric to F, Kannan's fixed point theorem yields a unique fixed point (p^*,q^*) of F and therefore a unique fixed point $u^*\in X_2$ of T.

Moreover, $(p,q)=F(p,q)$ is equivalent to $p=f(p)$ and $q=f(q)$. Since f has the unique fixed point 0, we obtain $(p^*,q^*)=(0,0)$ and $u^*=0$ a.e. on Ω. □

If T were a Banach contraction on $L^\infty(\Omega)$, then its restriction to the invariant complete subspace X_2 would also be a Banach contraction with the same constant, which is impossible because F is discontinuous. Hence T is not a Banach contraction on $L^\infty(\Omega)$.

V. Conclusion

Neural field operators often involve nonlinear or discontinuous activation functions that prevent the use of the classical Banach contraction principle. This raises the question of whether the existence and uniqueness of fixed points can still be established when the standard Lipschitz contraction condition fails.

The results of this paper show that the Kannan contraction framework provides a useful alternative in such situations. We first presented a discontinuous example illustrating that an operator may fail to be a Banach contraction while still satisfying a Kannan-type inequality. We then analyzed a neural field model with a nonlinear activation function and derived a Kannan-type estimate on a natural class of constant profiles. This example shows how residual terms may dominate state differences, although the corresponding class is not invariant under the operator in the present setting. Finally, for a neural field operator with Heaviside activation, we showed that the operator maps the space into a two-block subspace of $L^\infty(\Omega)$, reducing the problem to a finite-dimensional map on $\mathbb{R}^2$. The reduced map satisfies a Kannan-type inequality, which yields the existence and uniqueness of the fixed point.

These results indicate that Kannan-type estimates can extend fixed-point analysis for certain neural field operators that are discontinuous or not globally Lipschitz. Further work could investigate broader classes of neural field kernels and activation functions where similar conditions hold, as well as extensions to time-dependent neural field equations or higher-dimensional spatial domains.

ACKNOWLEDGMENTS

The authors acknowledge the financial support of the FCT in the framework of ARISE (DOI: 10.54499/LA/P/0112/2020) and R&D Unit SYSTEC (base UIDB/00147/2020 and programmatic UIDP/00147/2020 funds) and the project “Sound reconstruction and speech recognition based on a bio-inspired geometric model” ref. 2023.09597.CBM.

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