Optimal investment in a multi-asset market with borrowing, quadratic-affine interest rate, and the Heston volatility model

Nuha Alasmi[1,2] and Bujar Gashi[1]

***Abstract*— We consider the optimal investment problem in an incomplete multi-asset market with borrowing and random coefficients, under power utility from terminal wealth. The model combines the *Heston* stochastic volatility with a *general* class of quadratic-affine interest rate models, leading to a multi-input optimal stochastic control problem with a *nonlinear* system dynamics. By applying a certain *piece-wise* completion-of-squares method, we derive an explicit closed-form solution to this problem as a linear state-feedback control, the gain of which admits up to *five* different regimes.**

I. Introduction and Problem Formulation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which an m_1-dimensional standard Brownian motion $(W_1(t),\ t \geq 0)$ and an m_2-dimensional standard Brownian motion $(W_2(t),\ t \geq 0)$ are defined and assumed to be independent. We further consider the filtration $(\mathscr{F}(t),\ t \in [0,T])$, where $\mathscr{F}(t)$ is the augmentation of $\sigma\{W_1(s), W_2(s)\colon\ 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$. If E is an Euclidean space, then we denote by $L^\infty(0,T;E)$ the set of all E-valued uniformly bounded functions. The *optimal investment problem in a multi-asset market* with expected utility from terminal wealth in a continuous-time setting is formulated as follows. Consider a market of a *bond* with price B and n *stocks* with prices S_i, where $n \in \mathbb{N}$, and $i = 1,\ \ldots,\ n$, that are solutions to the following equations (for $i = 1,\ \ldots, n$, and $t \in [0,T]$):

$$\begin{cases} dB(t) = B(t)r(t)dt, \\ dS_i(t) = S_i(t)[\mu_i(t)dt + V_i'(t)dW_1(t)], \\ B(0) > 0, \quad S_i(0) > 0, \quad \text{are given.} \end{cases} \tag{1}$$

Here r is the interest rate, μ_i is the appreciation rate of S_i, and V_i is an m_1-dimensional volatility vector associated with S_i. The market coefficients r, μ_i, and V_i are random in general and such that equations (1) have unique strong solutions. Further, consider an *investor* with initial wealth $y_0 > 0$ that holds $v_B(t)$ number of shares in the bond and $v_{S_i}(t)$ number of shares in the stock i at time t. The value of investor's portfolio, i. e. the investor's wealth, at time t is thus:

$$y(t) := v_B(t)B(t) + \sum_{i=1}^n v_{S_i}(t)S_i(t), \quad t \in [0,T]. \tag{2}$$

This portfolio is called *self-financing* if it has the following dynamics (see, for example, [29], [27]):

$$dy(t) = v_B(t)dB(t) + \sum_{i=1}^n v_{S_i}(t)dS_i(t), \quad t \in [0,T]. \tag{3}$$

[1]Department of Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK. N.Alasmi@liverpool.ac.uk and Bujar.Gashi@liverpool.ac.uk

[2]Department of Mathematics, The University of Jeddah, KSA.

If we substitute the differentials of B and S_i from (1) into (3), and further knowing that $v_B(t)B(t) = y(t) - \sum_{i=1}^n v_{S_i}(t)S_i(t)$, which follows from (2), we obtain (for $t \in [0,T]$):

$$\begin{aligned} dy(t) &= \left[r(t)y(t) + \sum_{i=1}^n (\mu_i(t) - r(t))u_i(t)\right]dt \\ &+ \sum_{i=1}^n u_i(t)V_i'(t)dW_1(t), \end{aligned} \tag{4}$$

where $u_i(t) := v_{S_i}(t)S_i(t),\ i = 1,\ \ldots,\ n,\quad t \in [0,T]$. The self-financing portfolio (4) is thus an example of a *linear* stochastic control system with multiplicative noise with the investor's wealth y being the state of the system and the amounts of wealth invested in the stocks u_i being the control variables. The optimal investment problem with expected utility from terminal wealth as the criterion, is the following optimal stochastic control problem:

$$\begin{cases} \max\limits_{u(\cdot)\in\bar{\mathscr{Q}}} \mathbb{E}[U(y(T))], \\ \text{s.t. (4),} \end{cases} \tag{5}$$

for some suitable admissible set of controls $\bar{\mathscr{Q}}$ and utility function U. For a textbook account of problem (5), see, for example, [29], [27], [20]. Further, the special case of this problem with power utility $U(x) = x^\gamma/\gamma$ and $\gamma \in (0,1)$, admits an explicit closed-form solution in a linear state-feedback form (see, for example, [31], for the market with deterministic coefficients). Additionally, for the case of random and possibly *unbounded* coefficients see [23], and [3] for results on the random time horizon.

A more general and realistic market model than (1) is the one where the investor can *borrow* at a rate R that is higher than the bond rate r. In this case one restricts the *borrowed amount* $\Xi(t)$ to be (for $t \in [0,T]$):

$$\Xi(t) := [u'(t)\mathbf{1}_n - y(t)]^+ := \max[0, u'(t)\mathbf{1}_n - y(t)], \quad t \in [0,T],$$

where $u(t) := [u_1(t),\ \ldots,\ u_n(t)]'$, and $\mathbf{1}_n := [1,\ \ldots, 1]' \in \mathbb{R}^n$, as it is not reasonable to borrow with the higher rate R and at the same time invest in the bond with a lower rate r. The equation of investor's wealth (4) now becomes a *nonlinear* stochastic control system with multiplicative noise as follows:

$$\begin{aligned} &dy(t) = [r(t)y(t) + u'(t)(\mu(t) - r(t))]dt - (R(t) - r(t))\ \Xi dt \\ &+ u'(t)V(t)dW_1(t), \end{aligned} \tag{6}$$

where $\mu(t) := [\mu_1(t),\ \mu_2(t),\ \ldots,\ \mu_n(t)]'$,
$V(t) := [V_1'(t),\ V_2'(t),\ \ldots,\ V_n'(t)]$, $\quad t \in [0,T]$.

In the recent series papers [10], [11], [12], [13], [14], [1], [2], [3] several cases of the optimal investment problem in an *incomplete* market with borrowing, random interest rates, and the power utility have been considered, and it was succeeded in obtaining an explicit closed-form solution in the following cases: in [10] the interest rate is assumed to be quadratic-affine with independent source of uncertainty as compared to the stock; in [13] this was generalised further to permit for a general class of such coefficients; in [2] this was generalised to allow for a general class of such coefficients along with random time horizon instead of fixed horizon; in [12] the Hull-White model for the interest rate was used which has the same source of uncertainty as the stock; in [3] the Heston volatility model was used which has the same source of uncertainty as the stock; in [11] the market with a Markovian switching coefficients is considered; whereas in [14] and [1] markets with certain *combined* random interest rate models are introduced; [4] for a quadratic-affine interest rates and Heston stochastic volatility is considered. A common assumption in all of these papers on borrowing, is that an investor is only permitted to invest in a single stock. In [7], the optimal invetsment problem with borrowing for the case of *two stocks* and a certain class of coefficients is examined, whereas its multi-stock framework is developed in [8].

In this paper, we extend the framework of [4] by introducing a more general setting. Specifically, we generalise the optimal investment problem considered in [4] from a one-stock market to the multi-stock market. This makes the corresponding optimal investment problem and the derivation of its solution considerably more difficult to handle as compared to [4], which is particularly reflected in the form of the optimal control law (which now has five different regimes as compared to three such regimes in [4]). To solve this problem, our methodology is inspired by [8] and builds on a piecewise completion-of-squares method within a multi-dimensional framework. While the underlying approach is similar to that in [8], the present formulation is significantly more involved due to the nonlinearities arising from a stochastic interest rate incorporating a Heston-type and quadratic factor processes, coupled with stochastic volatility governed by the Heston model.

In order to formulate the model precisely, we introduce the following setting. First, let η denote the solution to the following nonlinear stochastic differential equation of Cox-Ingersoll-Ross (CIR) type (for $t \in [0,T]$):

$$\begin{cases} d\eta(t) = [\kappa(t)(\theta(t)-\eta(t))]dt + \sqrt{\eta(t)}\tilde{\sigma}'(t)dW_1(t), \\ \eta(0) = \eta_0 > 0, \quad \text{is given,} \end{cases} \tag{7}$$

where $0 < \kappa(\cdot) \in L^\infty(0,T;\mathbb{R})$, $0 < \theta(\cdot) \in L^\infty(0,T;\mathbb{R})$, $\tilde{\sigma}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_1})$, and $\tilde{\sigma}'(t)\tilde{\sigma}(t) > 0$ for all $t \in [0,T]$. The Heston volatility model is now defined as:

$$V(t) := \sigma(t)\sqrt{\eta(t)}, \quad t \in [0,T],$$

where $\sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m_1})$, and the stock volatility matrix σ is assumed to satisfy the following positivity property $\sigma(t)\sigma'(t) > 0$ for all $t \in [0,T]$, which ensures the absence of *arbitrage* (see, for example, [16]). The wealth equation (6) now becomes (for $t \in [0,T]$):

$$\begin{cases} dy = [ry + u'a - b[u'\mathbf{1}_n - y]^+]\,dt + u'\sigma\sqrt{\eta}dW_1, \quad t \in [0,T], \\ y(0) = y_0 > 0, \end{cases} \tag{8}$$

where a and b are defined as (for $t \in [0,T]$:

$a(t) = [a_1(t),\ \ldots,\ a_n(t)]' := [\mu_1(t)-r(t),\ \ldots,\ \mu_n(t)-r(t)]'$,
$b(t) := R(t) - r(t)$.

Moreover, we assume that the market coefficients a and b are linear functions of η defined as (for $t \in [0,T]$):

$$a(t) := \xi_1(t)\eta(t), \quad b(t) := \xi_2(t)\eta(t), \tag{9}$$

where $0 < \xi_1(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $0 < \xi_2(\cdot) \in L^\infty(0,T;\mathbb{R})$ (see, for example, [30] for a similar assumption on a). Note that our market is *incomplete* since the number of uncertainty sources (which in this case is $m_1 + m_2$, where $m_1 \geq n$) is in general larger than the number of stocks (which in this case is n). Further, we consider the following n-dimensional *factor process* x (for $t \in [0,T]$):

$$\begin{cases} dx(t) = [A_1(t)x(t) + A_0(t)]\,dt + B_0(t)dW_2(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given,} \end{cases} \tag{10}$$

where $A_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $A_0(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $B_0(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m_2})$. Further, we assume that the interest rate r is defined as (for $t \in [0,T]$):

$$r(t) := x'(t)M(t)x(t) + x'(t)N(t) + \phi(t)\eta(t) + Q(t), \tag{11}$$

where $M(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$ and symmetric, $N(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $\phi(\cdot) \in L^\infty(0,T;\mathbb{R})$, and $Q(\cdot) \in L^\infty(0,T;\mathbb{R})$. The model (11) with $\phi(t) = 0$, $t \in [0,T]$, is the well-known quadratic-affine interest rate model (see, for example, [30], [18], [24], [26], [1], [10], [12], [14], [9]), and by considering a not-necessarily zero ϕ, i.e. by permitting for the interest rate r to be influenced by the additional factor η, the model (11) is a generalisation of the quadratic-affine interest rate. The cost functional that we consider is the following power utility from terminal wealth:

$$J(u(\cdot)) := -\frac{1}{\gamma}\mathbb{E}\left[y^\gamma(T)\right], \quad \gamma \in (0,1). \tag{12}$$

which, due to its minus sign, is to be minimized. The corresponding optimal investment problem to be solved is the following multi-input optimal stochastic control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathscr{A}} J(u(\cdot)), \\ \text{s.t. } (8), \end{cases} \tag{13}$$

where $\mathscr{A}$ is a suitable set of admissible controls to be defined precisely later. In order to define $\mathscr{A}$ and state the solution to problem (13), we first define the following:

$$\xi(t) := (1-\gamma)\sigma(t)\sigma'(t).$$

For the (i,j)-th element of matrix ξ and its inverse ξ^{-1} we use the notation ξ_{ij} and $(\xi^{-1})_{ij}$, respectively. Since ξ is symmetric, it satisfies $\xi_{ij} = \xi_{ji}$ for all i,j, and similarly for

ξ^{-1}. Moreover, we introduce the following processes (for $t\in[0,T]$, $n\in\mathbb{N}$, and $\nu:=y^{-1}u$,):

$$\nu=\begin{bmatrix}\nu_1\\ \nu_2\\ \vdots\\ \nu_n\end{bmatrix}=\begin{bmatrix}1-(\nu_2+\cdots+\nu_n)\\ \nu_2\\ \vdots\\ \nu_n\end{bmatrix}=\begin{bmatrix}1-\mathbf{1}_{n-1}\underline{\nu}\\ \underline{\nu}\end{bmatrix},$$

where

$$\underline{\nu}:=\begin{bmatrix}\nu_2\\ \nu_3\\ \vdots\\ \nu_n\end{bmatrix},\ \alpha_0:=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix},\ \beta=\xi_1+\bar{\phi}\sigma\tilde{\sigma}', \tag{14}$$

$$\alpha_1:=\begin{pmatrix}-1 & -1 & -1 & \cdots & -1\\ 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix},\ n_1:=\left(\xi^{-1}\beta\right)'\mathbf{1}_n,$$

$$G(\nu):=\frac{1}{2}\nu'\xi\nu-\nu'\beta+\xi_2[\nu'\mathbf{1}_n-1]^+,$$

$$n_2:=\left(\xi^{-1}\beta-\xi^{-1}\xi_2\right)'\mathbf{1}_n,\quad d_1=\xi^{-1}\beta,\quad d_2:\xi^{-1}\left(\beta-\xi_2\mathbf{1}_n\right),$$

$$d_3:=\alpha_0+\bar{g}^{-1}\underline{g}',\quad\text{where}\quad\bar{g}:=\alpha_1'\xi\alpha_1,$$

$$\underline{g}=\left(\alpha_1'\beta-\alpha_1'\xi\alpha_0\right),\ c_1:=-\frac{1}{2}\beta'\xi^{-1}\beta,$$

$$c_2:=-\frac{1}{2}\left(\beta-\xi_2\mathbf{1}_n\right)'\xi^{-1}\left(\beta-\xi_2\mathbf{1}_n\right)-\xi_2,$$

$$c_3:=\frac{1}{2}[-\underline{g}'\bar{g}^{-1}\underline{g}+\alpha_0'\xi\alpha_0-2\alpha_0'\beta].$$

Secondly, we present the assumptions and integrability requirements under which problem (13) admits a solution. Therefore, to define the admissible set of controls $\mathscr{A}$ we begin by introducing the following nonlinear backward ordinary differential equation:

$$\begin{cases}\dot{\bar{\phi}}-\bar{\phi}\kappa+\frac{1}{2}\bar{\phi}^2\tilde{\sigma}'\tilde{\sigma}+\gamma\phi-\gamma\Psi=0,\\ \bar{\phi}(T)=0,\end{cases} \tag{15}$$

Here Ψ is a certain function of $\bar{\phi}$ defined as ($t\in[0,T]$):

$$\Psi(t):=\begin{cases}c_1 & \text{if } n_1\le 1 \text{ and } n_2\le 1,\\ c_1 & \text{if } n_1\le 1 \text{ and } n_2>1,\\ c_2 & \text{if } n_1\le 1 \text{ and } n_2>1,\\ c_2 & \text{if } n_1>1 \text{ and } n_2>1,\\ c_3 & \text{if } n_1>1 \text{ and } n_2\le 1.\end{cases}$$

As β, which is defined in (14), is a linear function of $\bar{\phi}$, equation (15) has at most quadratic growth in $\bar{\phi}$.

Assumption 1. *The nonlinear backward ordinary differential equation (15) has a unique global solution.*

In the special case of $\sigma'(t)\tilde{\sigma}(t)=0$ for $t\in[0,T]$, equation (15) reduce to a Riccati backward differential equation (see, e. g., [34] and [35], for sufficient conditions under which Assumption 1 holds). Further consider the following *Riccati* backward ordinary differential equation (for $t\in[0,T]$):

$$\begin{cases}\dot{\bar{M}}+A_1'\bar{M}+\bar{M}A_1+2\bar{M}B_0B_0'\bar{M}+\gamma M=0,\\ \bar{M}(T)=0,\end{cases} \tag{16}$$

Assumption 2. *The Riccati backward ordinary differential equation (16) has a unique global solution.*

Sufficient conditions for this assumption to hold can be derived from the results in [34] and [35]. As an example, if $M(t)=0$ for $t\in[0,T]$, then by inspection we conclude that $\bar{M}(t)=0$, $t\in[0,T]$, is the unique solution to (16). Furthermore, we introduce the following backward ordinary differential equations (for $t\in[0,T]$):

$$\begin{cases}\dot{\bar{N}}+2\bar{M}A_0+A_1'\bar{N}+2\bar{M}B_0B_0'\bar{N}'+\gamma N=0,\\ \bar{N}(T)=0,\end{cases} \tag{17}$$

$$\begin{cases}\dot{p}+\frac{1}{2}p\,\mathrm{tr}\left(B_0'\bar{M}B_0\right)+pA_0'\bar{N}+p\bar{\phi}\kappa\theta+\frac{1}{2}p\bar{N}B_0B_0'\bar{N}'+\gamma pQ=0,\\ p(T)=-\frac{1}{\gamma},\end{cases} \tag{18}$$

As (17) and (18) are linear in $\bar{N}$ and p, respectively, our Assumption 1 and Assumption 2, ensure the existence of unique global solutions to these equations. To define the admissible set of controls $\mathscr{A}$, we introduce the function ρ as (for $t\in[0,T]$):

$$\rho(t):=\begin{cases}d_1 & \text{if } n_1\le 1 \text{ and } n_2\le 1,\\ d_1 & \text{if } c_1\le c_2,\ n_1\le 1 \text{ and } n_2>1,\\ d_2 & \text{if } c_1>c_2,\ n_1\le 1 \text{ and } n_2>1,\\ d_2 & \text{if } n_1>1 \text{ and } n_2>1,\\ d_3 & \text{if } n_1>1 \text{ and } n_2\le 1,\end{cases}$$

The admissible set of controls $\mathscr{A}$ is defined as the set of all $\mathbb{R}^n$-valued adapted processes u under which the wealth equation (8) has a unique and strictly positive strong solution, i.e. $y(t)>0$ *a.s.* for all $t\in[0,T]$, and the following integrability conditions hold:

$$\mathbb{E}\left[\int_0^T pe^Fy^\gamma\sqrt{\eta}\left(\gamma\frac{u}{y}\sigma'+\bar{\phi}\tilde{\sigma}'\right)dW_1\right]=0, \tag{19}$$

$$\mathbb{E}\left[\int_0^T pe^Fy^\gamma\left(2x'\bar{M}+\bar{N}'\right)B_0dW_2\right]=0. \tag{20}$$

where F is defined in a quadratic–affine form with respect to x and linearly with respect to η (for $t\in[0,T]$):

$$F(t)=x'(t)\bar{M}(t)x(t)+x'(t)\bar{N}(t)+\bar{\phi}(t)\eta(t), \tag{21}$$

here $\bar{M}$ is symmetric. The strict positivity requirement on the wealth y avoids investor's bankruptcy, whereas the above integrability conditions ensures that certain stochastic integrals appearing in the proof of Theorem 2.2 have a zero expectation. This completes the formulation of problem (13) to be solved below. The expected value and the variance-covariance matrix of the factor process (10) are denoted as (for $t\in[0,T]$) :

$$\mu_x(t)\ :=\ \mathbb{E}[x(t)],\quad \Sigma_x(t)=\mathbb{E}[(x(t)-\mu_x(t))(x(t)-\mu_x(t))'].$$

These are solutions to the following forward differential equations (see, for example, [28]):

$$\dot{\mu}_x\ =\ A_1\mu_x+A_0,\quad \mu_x(0)=x_0,$$

$$\dot{\Sigma}_x=A_1\Sigma_x+\Sigma_xA_1'+B_0B_0',\quad \Sigma_x(0)=0.$$

Assumption 3. *Let $q_1,\ q_2,\ p_1,\ p_2,\ n_1,\ n_2,\ h_1, h_2 > 1$ be such that $q_1^{-1}+q_2^{-1}=1,\ p_1^{-1}+p_2^{-1}=1,\ n_1^{-1}+n_2^{-1}=1$, and ${h_1}^{-1}+{h_2}^{-1}=1$. The following hold (for $t\in[0,T]$):*
(i) $\Sigma_x(t)>0,\ \Sigma_x^{-1}(t)-32\ \bar{M}(t)>0,$

(ii) $4\kappa(t)\theta(t)\le\tilde{\sigma}'(t)\tilde{\sigma}(t),$

(iii) $Z(t)t\displaystyle\int_0^t e^{\int_0^s\kappa(\tau)d\tau}\tilde{\sigma}'(s)\tilde{\sigma}(s)ds<2,$

where $Z(t):=\sup\limits_{s\in[0,t]}[n_2(-16\bar{\phi}\kappa+128\bar{\phi}^2 n_1\tilde{\sigma}'\tilde{\sigma})],$

(iv) $H(t)t\displaystyle\int_0^t e^{\int_0^s\kappa(\tau)d\tau}\tilde{\sigma}'(s)\tilde{\sigma}(s)ds<2,$

where $H(t){:=}\sup\limits_{s\in[0,t]}h(s),\quad t\in[0,T],$

with $C(t):=\phi+\Psi'\xi_1-\xi_2[\Psi'\mathbf{1}_n-1]^+-\frac{1}{2}\Psi'\sigma\sigma'\Psi,$ and $h(t):=8\gamma p_2 q_2(C+4\gamma q_1 p_2\Psi'\sigma\sigma'\Psi).$

(v) $\mathbb{E}\left[\exp\left(8\gamma p_1\displaystyle\int_0^t\hat{F}ds\right)\right]<\infty,$

where $\hat{F}(t):=x'(t)M(t)x(t)+x'(t)N(t)+Q(t),$

(vi) $L(t)t\displaystyle\int_0^t e^{\int_0^s\kappa(\tau)d\tau}\tilde{\sigma}'(s)\tilde{\sigma}(s)ds<2,$ where

$L(t):=\sup_{s\in[0,t]}\ell(s),\quad t\in[0,T],\quad$ with $\ell(t):=h_2(-\lambda\kappa+\lambda^2 h_1\tilde{\sigma}'\tilde{\sigma}/2).$

All of the above assumption are in terms of known functions and processes, and can thus be verified for each concrete market example.

II. Solution to the optimal investment problem

To present the solution to the optimal control problem (13), we first introduce the following theorem.

Theorem 2.1: $\min\limits_{\nu\in\mathbb{R}^n}G(\nu)=\Psi$, and the corresponding minimizer is $\nu^*:=\rho$.

Proof: By a piece-wise completion of squares, we write $G(\nu)$ as a piece-wise quadratic function, obtained using the definitions of a and b as given in (9):

$$\begin{aligned}G(\nu) &= \frac{1}{2}\left[\nu'\xi\nu-2\nu'\beta+2\xi_2[\nu'\mathbf{1}_n-1]^+\right]\\ &= \frac{1}{2}\left[\left(\nu-\xi^{-1}\beta\right)'\xi\left(\nu-\xi^{-1}\beta\right)-\beta'\xi^{-1}\beta\right]I_{(\nu'\mathbf{1}_n\le1)}\\ &+ \frac{1}{2}\bigg\{\left[\nu-\xi^{-1}(\beta-\xi_2\mathbf{1}_n)\right]'\xi\left[\nu-\xi^{-1}(\beta-\xi_2\mathbf{1}_n)\right]\\ &- (\beta-\xi_2\mathbf{1}_n)'\xi^{-1}(\beta-\xi_2\mathbf{1}_n)-2\xi_2\bigg\}I_{(\nu'\mathbf{1}_n>1)}.\end{aligned}\tag{22}$$

where $I_{(\cdot)}$ is the indicator function. The minimizer of $G(\nu)$ and its corresponding minimum value are determined based on the following cases:

Case (1): if $n_1\le1$ and $n_2\le1$, then $\nu^*=d_1$ and $G(\nu^*)=c_1$.
Case (2): if $c_1\le c_2,\ n_1\le1$, and $n_2>1$, then $\nu^*=d_1$ and $G(\nu^*)=c_1$.
Case (3): if $c_1>c_2,\ n_1\le1$, and $n_2>1$, then $\nu^*=d_2$ and $G(\nu^*)=c_2$.
Case (4): if $n_1>1$ and $n_2>1$, then $\nu^*=d_2$ and $G(\nu^*)=c_2$.
Case (5): if $n_1>1$ and $n_2\le1$, then $\nu^*=d_3$ and $G(\nu^*)=c_3$.

The derivation of the minimiser of $G(\nu)$ and its corresponding minimum value for cases (1)–(4) follows directly from the imposed conditions and interval constraints. We now turn to case (5), where we characterise the minimiser ν^* and the associated value $G(\nu^*)$. This case corresponds to a vector $\nu=[\nu_1\ \nu_2\ \cdots\ \nu_n]'$ satisfying the following condition:

$$\nu_1+\nu_2+\cdots+\nu_n=1.\tag{23}$$

Next, we express ν in terms of $\underline{\nu}$ as

$$\nu:=\alpha_0+\alpha_1\underline{\nu},\tag{24}$$

where $\underline{\nu}\in\mathbb{R}^{n-1}$. Substituting (24) into (22), we rewrite $G(\nu)$ as a function of $\underline{\nu}$. This transformation reduces the problem to a problem in $\underline{\nu}$, allowing us to derive the minimiser and compute the corresponding minimum value. By substituting $\underline{\nu}^*$ into (24), we obtain the corresponding minimiser ν^*, and the minimum value as follows:

$$G(\nu)=\frac{1}{2}\left[\left(\nu-\xi^{-1}\beta\right)'\xi\left(\nu-\xi^{-1}\beta\right)-\beta'\xi^{-1}\beta\right]I_{(\nu'\mathbf{1}\le1)}\tag{25}$$

$$\begin{aligned}&=\frac{1}{2}\bigg\{\left[(\alpha_0+\alpha_1\underline{\nu})-\xi^{-1}\beta\right]'\xi\left[(\alpha_0+\alpha_1\underline{\nu})-\xi^{-1}\beta\right]\\ &-\beta'\xi^{-1}\beta\bigg\}=\frac{1}{2}\Big[(\alpha_0+\alpha_1\underline{\nu})'\xi(\alpha_0+\alpha_1\underline{\nu})-(\alpha_0+\alpha_1\underline{\nu})'\\ &\times\beta-\beta'(\alpha_0+\alpha_1\underline{\nu})+\beta'\xi^{-1}\beta-\beta'\xi^{-1}\beta\Big]\\ &=\frac{1}{2}\Big[\alpha_0'\xi\alpha_0+\alpha_0'\xi\alpha_1\underline{\nu}+\underline{\nu}'\alpha_1'\xi\alpha_0+\underline{\nu}'\alpha_1'\xi\alpha_1\underline{\nu}-\alpha_0'\beta\\ &-\underline{\nu}'\alpha_1'\beta-\beta'\alpha_0-\beta'\alpha_1\underline{\nu}\Big]\\ &=\frac{1}{2}\left[\underline{\nu}'\alpha_1'\xi\alpha_1\underline{\nu}+2\underline{\nu}'\alpha_1'\xi\alpha_0-2\underline{\nu}'\alpha_1'\beta+\alpha_0'\xi\alpha_0-2\alpha_0'\beta\right]\\ &=\frac{1}{2}\left[\underline{\nu}'\bar{g}\underline{\nu}-2\underline{\nu}'\underline{g}+\alpha_0'\xi\alpha_0-2\alpha_0'\beta\right]\\ &=\frac{1}{2}\left\{\left[\underline{\nu}-\bar{g}^{-1}\underline{g}\right]'\bar{g}\left[\underline{\nu}-\bar{g}^{-1}\underline{g}\right]-\underline{g}'\bar{g}^{-1}\underline{g}+\alpha_0'\xi\alpha_0-2\alpha_0'\beta\right\}.\end{aligned}$$

Therefore, the minimizer of $G(\nu)$ is:

$$\nu^* \quad=\quad d_3=\alpha_0+\alpha_1\bar{g}^{-1}\underline{g}'.\tag{26}$$

and the minimum value of $G(\nu)$ is:

$$G(\nu^*)=c_3=\frac{1}{2}\left[-\underline{g}'\bar{g}^{-1}\underline{g}+\alpha_0'\xi\alpha_0-2\alpha_0'\beta\right].\qquad\square$$

We can now give the solution to the optimal control problem (13):

Theorem 2.2: There exists a unique solution to the optimal investment problem (13). This solution is given as:

$$u^*(t)=\rho(t)y(t),\qquad t\in[0,T].$$

The corresponding optimal cost functional is:

$$J\left(u^*(\cdot)\right)=y_0^{\gamma}p(0)e^{x_0'\bar{M}(0)x_0+x_0'\bar{N}(0)+\bar{\phi}(0)\eta_0}.$$

Proof. By Itô's formula, the differential of y^γ is:

$$\begin{aligned}dy^\gamma=&\bigg\{\gamma y^{\gamma-1}\left[ry+u'a-b\left[u'\mathbf{1}_n-y\right]^+\right]\\ &+\frac{(\gamma-1)\gamma}{2}y^{\gamma-2}u'\sigma\sigma'u\bigg\}dt+\gamma y^{\gamma-1}\sqrt{\eta}u'\sigma dW_1.\end{aligned}$$

The differential of F by Itô's formula is:

$$dF=(A_1xdt+A_0)'\bar{M}x(t)dt+(B_0dW_2)'\bar{M}x+x'\dot{\bar{M}}xdt+x'\bar{M}(A_1 \times x+A_0)dt+x'\bar{M}B_0dW_2+\frac{1}{2}\ tr\left(B_0'\bar{M}B_0\right)dt+(A_1x+A_0)'\bar{N}dt + (B_0dW_2)'\bar{N}+x'\dot{\bar{N}}dt+\dot{\bar{\phi}}\eta dt+\bar{\phi}\kappa(\theta-\eta)dt+\bar{\phi}\sqrt{\eta}\tilde{\sigma}'dW_1$$
$$=\left\{x'\left(A_1'\bar{M}+\bar{M}A_1+\dot{\bar{M}}\right)x+x'\left(2\bar{M}A_0+\dot{\bar{N}}+A_1'\bar{N}\right) + \frac{1}{2}\mathrm{tr}\left(B_0'\bar{M}B_0\right)+A_0'\bar{N}+\eta(\dot{\bar{\phi}}-\bar{\phi}\kappa)+\bar{\phi}\kappa\theta\right\}dt+2x'\bar{M}B_0dW_2 +dW_2'B_0'\bar{N}+\bar{\phi}\sqrt{\eta}\tilde{\sigma}'dW_1.$$

Further, by Itô's formula, the differential of Pe^F is:

$$d\left(pe^F\right)=\dot{p}e^Fdt+pe^F\left[x'\left(A_1'\bar{M}+\bar{M}A_1+\dot{\bar{M}}\right)x+x'\left(2\bar{M}A_0 +\dot{\bar{N}}+A_1'\bar{N}\right)+\tfrac{1}{2}tr\left(B_0'\bar{M}B_0\right)+A_0'\bar{N}+\eta(\dot{\bar{\phi}}-\bar{\phi}\kappa)+\bar{\phi}\kappa\theta\right]dt + pe^F\left[\bar{\phi}\sqrt{\eta}\tilde{\sigma}'dW_1+\left(2x'\bar{M}+\bar{N}\right)'B_0dW_2\right]+\tfrac{1}{2}pe^F\bar{\phi}^2\tilde{\sigma}'\tilde{\sigma}\eta dt + \tfrac{1}{2}pe^F\left(2x'\bar{M}+\bar{N}\right)B_0B_0'\left(2x'\bar{M}+\bar{N}\right)'dt=\left\{\dot{p}e^F+pe^F\left[x'\left(A_1' \times\bar{M}+\bar{M}A_1+\dot{\bar{M}}+2\bar{M}B_0B_0'\bar{M}x+\tfrac{1}{2}\ tr\left(B_0'\bar{M}B_0\right)+x'\left(2\bar{M}A_0+\dot{\bar{N}} +A_1'\bar{N}+2\bar{M}B_0B_0'\bar{N}'\right)+A_0'\bar{N}+\bar{\phi}\kappa\theta+\tfrac{1}{2}\bar{N}B_0B_0'\bar{N}'+\eta\left(\dot{\bar{\phi}}-\bar{\phi}\kappa +\tfrac{1}{2}\bar{\phi}^2\tilde{\sigma}'\tilde{\sigma}\right)\right]\right\}dt+pe^F\left[\left(2x'\bar{M}+\bar{N}'\right)B_0dW_2+\bar{\phi}\sqrt{\eta}\tilde{\sigma}'dW_1\right].$$

Now, by Ito's product rule, the differential of $y^\gamma Pe^F$ is:

$$d\left(y^\gamma pe^F\right)=(dy^\gamma)pe^F+y^\gamma d\left(pe^F\right)+(dy^\gamma)\left(dpe^F\right)$$
$$=\left[\gamma y^{\gamma-1}\left(ry+u'a-b\left[u'\mathbf{1}_n-y\right]^+\right)+\tfrac{\gamma(\gamma-1)}{2}y^{\gamma-2}\eta u'\sigma\sigma'u\right] \times pe^Fdt+\gamma y^{\gamma-1}\sqrt{\eta}u'\sigma pe^FdW_1+y^\gamma\left\{\dot{p}e^F+pe^F\left[x'\left(A_1'\bar{M} +\bar{M}A_1+\dot{\bar{M}}+2\bar{M}B_0B_0'\bar{M}\right)x+x'\left(2\bar{M}A_0+\dot{\bar{N}}+A_1'\bar{N}+2\bar{M}B_0 \times B_0'\bar{N}'\right)+\tfrac{1}{2}\mathrm{tr}\left(B_0'\bar{M}B_0\right)+A_0'\bar{N}+\bar{\phi}\kappa\theta+\tfrac{1}{2}\bar{N}B_0B_0'\bar{N}'+\eta\left(\dot{\bar{\phi}} -\bar{\phi}\kappa\tfrac{1}{2}\phi^2\tilde{\sigma}'\tilde{\sigma}\right)\right]\right\}dt+py^\gamma e^F\left(2x'\bar{M}+\bar{N}'\right)B_0dW_2+py^\gamma e^F\bar{\phi} \times\sqrt{\eta}\tilde{\sigma}'dW_1+\gamma\bar{\phi}\eta u'\sigma\tilde{\sigma}'y^{\gamma-1}dt.$$

If we define $\nu(t):=\frac{u(t)}{y(t)},\ \forall t\in[0,T]$, then:

$$d\left(y^\gamma pe^F\right)=\left\{pe^F\gamma y^\gamma\left[x'Mx+x'N+Q+\phi\eta+\nu'a -b\left[\nu'\mathbf{1}_n-1\right]^++\tfrac{1}{2}(\gamma-1)\eta\nu'\sigma\sigma'\nu\right]+y^\gamma e^F\left[\dot{p}+p\left(x'\left(A_1'\bar{M} +\bar{M}A_1+\dot{\bar{M}}+2\bar{M}B_0B_0'\bar{M}\right)x+x'\left(2\bar{M}A_0+\dot{\bar{N}}+A_1'\bar{N} +2\bar{M}B_0B_0'\bar{N}'\right)+\tfrac{1}{2}\mathrm{tr}\left(B_0'\bar{M}B_0\right)+A_0'\bar{N}+\bar{\phi}\kappa\theta+\tfrac{1}{2}\bar{N}B_0B_0'\bar{N}' +\eta\left(\dot{\bar{\phi}}-\bar{\phi}\kappa+\tfrac{1}{2}\phi^2\tilde{\sigma}'\tilde{\sigma}+\phi\right)+\gamma\bar{\phi}\eta\nu'\sigma\tilde{\sigma}'\right)\right]\right\}dt+py^\gamma e^F \times\left(2x'\bar{M}+\bar{N}'\right)B_0dW_2+py^\gamma e^F\bar{\phi}\sqrt{\eta}\tilde{\sigma}'dW_1+\gamma y^\gamma\sqrt{\eta}\nu'\sigma pe^FdW_1$$

By integrating both sides of the above equation from 0 to T, and then taking the expectation, we obtain the following for any admissible control (note that due to the integrability requirements (19) and (20) the expectation of stochastic integrals are zero):

$$-\frac{1}{\gamma}\mathbb{E}\left[y^\gamma(T)\right]=y_0^\gamma p_0e^{F_0}+\mathbb{E}\left\{\int_0^T\left\{y^\gamma pe^F\left[x'(t)\left(A_1'\bar{M}+\bar{M}A_1 +\dot{\bar{M}}+2\bar{M}B_0B_0'\bar{M}+\gamma M\right)x(t)+x'(t)\left(2\bar{M}A_0+\dot{\bar{N}}+A_1'\bar{N}+2\bar{M} \times B_0B_0'\bar{N}'+\gamma N\right)+A_0'\bar{N}+\gamma Q+\tfrac{1}{2}tr(B_0'\bar{M}B_0)+\bar{\phi}\kappa\theta+\frac{1}{2}\bar{N}B_0 \times B_0'\bar{N}'+\eta\left(\dot{\bar{\phi}}-\bar{\phi}\kappa+\tfrac{1}{2}\phi^2\tilde{\sigma}'\tilde{\sigma}+\gamma\phi\right)\right]+e^Fy^\gamma\left[\dot{p}+\gamma p\left(\nu'\xi_1 \times\eta-\xi_2\eta\left[\nu'\mathbf{1}_n-1\right]^++\tfrac{1}{2}(\gamma-1)\eta\nu'\sigma\sigma'\nu+\bar{\phi}\nu'\sigma\tilde{\sigma}'\right)\right]\right\}dt\right\}.$$

The terms of the integrand in the above representation of J that depend on ν are defined as in (25). Therefore, an integral representation of J for all admissible controls is thus:

$$J(u(\cdot))=y_0^\gamma p_0e^{F_0}+\mathbb{E}\left\{\int_0^T y^\gamma e^F\left[p\eta\left(\dot{\bar{\phi}}-\bar{\phi}k+\tfrac{1}{2}\bar{\phi}^2\tilde{\sigma}'\tilde{\sigma}+\gamma\phi +\gamma p\left(\nu'\xi_1-\xi_2[\nu'\mathbf{1}_n-1]^++\tfrac{1}{2}(\gamma-1)\nu'\sigma\sigma'\nu+\bar{\phi}\nu'\sigma'\tilde{\sigma}\right)\right)\right]dt\right\}$$
$$=y_0^\gamma p_0e^{F_0}+\mathbb{E}\left\{\int_0^T y^\gamma e^F\left[\eta p\left(\dot{\bar{\phi}}-\bar{\phi}k+\tfrac{1}{2}\bar{\phi}^2\tilde{\sigma}'\tilde{\sigma}+\gamma\phi-\gamma\Psi\right) -\gamma p\left(G(\nu)-\Psi\right)\right]dt\right\}, \tag{27}$$

where $y(0)=y_0$, $p(0)=p_0$, and $F(0)=F_0$. It then follows from Theorem 2.1 that $\min_\nu G(\nu)=\Psi$, and the corresponding minimizer is $\nu^*=\rho$. Consequently, for any admissible control $u(\cdot)$, the cost functional (27) satisfies

$$J(u(\cdot))=y_0^\gamma p_0e^{F_0}-\mathbb{E}\left\{\int_0^T y^\gamma pe^F\gamma\left(G(\nu)-\Psi\right)dt\right\} \geq y_0^\gamma p_0e^{F_0}.$$

This lower bound is achieved if and only if $\nu^*(t)=\nu(t)=\rho(t)$ *a.e.* $t\in[0,T]$ *a.s.*, or equivalently, if and only $u^*(t)=\rho(t)y(t)$ *a.e.* $t\in[0,T]$ *a.s.*. The corresponding optimal cost functional is thus $J(u^*(\cdot))=y_0^\gamma p_0e^{F_0}$. To show that $u^*(\cdot)\in\mathscr{A}$, we first introduce the following processes for $t\in[0,T]$:

$$\begin{aligned}\Pi(t) &:= r(t)+\rho'(t)a(t)-b(t)[\rho'(t)\mathbf{1}_n-1]^+,\\ &= x'(t)M(t)x(t)+x'(t)N(t)+Q(t)+\phi(t)\eta(t)\\ &+ \rho'(t)\xi_1(t)\eta(t)-\xi_2(t)\eta(t)[\rho'(t)\mathbf{1}_n-1]^+,\\ \Sigma(t) &:= \rho'(t)\sigma(t)\sqrt{\eta(t)},\\ Y(t) &:= y_0\left\{\exp\left[\int_0^t\left(\Pi(s)-\Sigma(s)\Sigma'(s)/2\right)ds\right.\right.\\ &+ \left.\left.\int_0^t\Sigma'(s)dW_1(s)\right]\right\}.\end{aligned} \tag{28}$$

As u^* is of linear state-feedback form and $\rho(\cdot)\in L^\infty(0,T;\mathbb{R}^n)$, it follows (in the same way as in the proof of Lemma 3.1 of [1]) that under u^* the wealth equation (8) has a unique strong solution given explicitly as $y(t)=Y(t)$, which in particular means that $y(t)>0$ *a.s.* for all $t\in[0,T]$. Moreover, the requirements (19) and (20) hold if the integrands appearing there are square-integrable processes. It is thus sufficient to show that:

$$\mathbb{E}\left[e^{2F}y^{2\gamma}x'(t)x(t)\right]<\infty,\quad t\in[0,T], \tag{29}$$
$$\mathbb{E}\left[e^{2F}y^{2\gamma}\eta(t)\right]<\infty,\quad t\in[0,T]. \tag{30}$$

By applying Lemma 2.2 in [4] and Assumption 3 (i)–(vi), the finiteness of (29) and (30) follows by an argument analogous to that in [4]. Therefore, the control $u^*(\cdot)$ defined in Theorem 2.2 belongs to the admissible set $\mathscr{A}$. $\square$

III. Conclusion

We have studied the optimal investment problem in a multi-asset market with borrowing, where the model involves random (possibly unbounded) coefficients. The volatility is governed by a Heston-type model, while the interest rate follows a generalized quadratic-affine term structure. The objective is to maximize a power utility from terminal wealth. This leads to a nonlinear stochastic optimal control problem with unbounded coefficients. An explicit closed-form solution is derived in the form of a linear wealth-feedback control, the gain of which lies in one of up to five regimes (see the definitions of ρ and u^*). This provides a rare example of a multi-input nonlinear stochastic control problem that admits a fully explicit solution. In this context, several additional directions of interest may be explored. For instance, incorporating consumption into the investor's wealth model would significantly increase the complexity of the problem. Similarly, introducing additional sources of market randomness, such as Markovian switching coefficients (see, e.g., [11]), which model the sudden jumps in the market, would further widen the applicability of the results.

References

[1] N. Alasmi and B. Gashi, Optimal investment in a market with borrowing and combined interest rate, *9th International Conference on Control, Decision, and Information Technologies (CoDIT'23)*, (2023), 2550-2555.

[2] N. Alasmi and B. Gashi, Optimal investment in a market with borrowing, unbounded random coefficients, and a random time-horizon, *2024 UKACC 14th International Conference on Control (CONTROL)*, (2024), 248-253.

[3] N. Alasmi and B. Gashi, Optimal investment in a market with borrowing and the Heston volatility model, *Proceedings of the 10th International Conference on Control, Decision and Information Technologies (CoDIT)*, (2024), 848-853.

[4] N. Alasmi and B. Gashi, Optimal investment in a market with borrowing, quadratic-affine interest rates, and Heston stochastic volatility, *Memorias del Congreso Nacional de Control Automático*, (2024), 262-267.

[5] N. Alasmi and B. Gashi, Optimal Investment in a Market with Borrowing, the CIR Interest Rate Model, and the Heston Volatility Model, *IFAC-PapersOnLine*, 58 (2024), 55-60.

[6] N. Alasmi, N and B. Gashi, Optimal investment in a market with borrowing, quadratic-affine interest rates, and Heston stochastic volatility. *Memorias del Congreso Nacional de Control Automático*, 7 (2024), 262-267.

[7] N. Alasmi and B. Gashi, Optimal investment in a multi-asset market with borrowing and unbounded random coefficients. *Proceedings of the 11th International Conference on Control, Decision and Information Technologies (CoDIT)*, 1 (2025), 2358-2363. IEEE.

[8] N. Alasmi and B. Gashi, Optimal investment in a multi-asset market with borrowing and unbounded random coefficients, *IFAC-PapersOnLine*, 59 (2025), 82-87,

[9] M. Algoulity and B. Gashi, Optimal regulator for a class of nonlinear stochastic systems with random coefficients, *European Journal of Control*, (2023), 100844.

[10] A. Aljalal and B. Gashi, Optimal investment in a market with borrowing and quadratic-affine interest rates, *European Control Conference (ECC)*, (2022), 1146-1151.

[11] A. Aljalal and B. Gashi, Optimal investment and consumption in a market with Markovian switching coefficients and borrowing, *17th International Conference on Control and Automation (ICCA)*, (2022), 166-171.

[12] A. Aljalal and B. Gashi, Optimal investment in a market with random interest rate for borrowing: an explicit closed-form solution, *8th International Conference on Control, Decision, and Information Technologies (CoDIT'22)*, (2022), 764-769.

[13] A. Aljalal and B. Gashi, Optimal investment in a market with borrowing and unbounded random coefficients, *2022 UKACC 13th International Conference on Control (CONTROL)*, (2022), 142-147.

[14] A. Aljalal and B. Gashi, Optimal investment in a market with borrowing, unbounded random coefficients, and a combined interest rate model, *European Control Conference (ECC)*, (2023), 143-147.

[15] L. Bergomi, *Stochastic volatility modelling*, CRC Press, 2015.

[16] N. H. Bingham and R. Kiesel, *Risk-neutral valuation*, Springer, 2004.

[17] J. Cvitanić and I. Karatzas, Convex duality in constrained portfolio optimization, *The Annals of Applied Probability*, 2 (1992), 767-818.

[18] P. Date and B. Gashi, Risk-sensitive control for a class of nonlinear systems with multiplicative noise, *Systems & Control Letters*, 62 (2013), 989-999.

[19] S. Desmettre, R. Korn, and T. Sayer, Option pricing in practice-Heston's stochastic volatility model, *Currents in Industrial Mathematics*, (2015), 351-400.

[20] D. Duffie, *Dynamic asset pricing theory*, Princeton University Press, 2010.

[21] W. Fleming and T. Zariphopoulou, An optimal investment/consumption model with borrowing, *Mathematics of Operations Research*, 16 (1991), 802-822.

[22] L. I. Gal'chuk, Existence and uniqueness of solution for stochastic equations with respect to semimartingales, *Theory of Probability & Its Applications*, 23 (1979), 751-763.

[23] B. Gashi and J. Li, Integrability of exponential process and its application to backward stochastic differential equations, *IMA Journal of Management Mathematics*, 30 (2019), 335-365.

[24] B. Gashi and M. Zhang, Indefinite risk-sensitive control, *E. J. of Control*, 69 (2023), 100741.

[25] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The Review of Financial Studies*, 6 (1993), 327-343.

[26] H. Hua, B. Gashi, and M. Zhang, Robust risk-sensitive control, *International Journal of Robust and Nonlinear Control*, 33 (2023), 5177-5868.

[27] I. Karatzas and S. E. Shreve, *Methods of mathematical finance*, Springer, 1998.

[28] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer, 1992.

[29] R. Korn, *Optimal portfolios: stochastic models for optimal investment and risk management in continuous time*, World Scientific, 1997.

[30] J. Liu, Portfolio selection in stochastic environments, *The Review of Financial Studies*, 20 (2007), 1-39.

[31] R. C. Merton, *Continuous-time finance*, Wiley-Blackwell, 1992.

[32] S. Mikhailov, and U. Nögel, *Heston's stochastic volatility model: implementation, calibration and some extensions*, John Wiley & Sons, 2004.

[33] M. Musiela and M. Rutkowski, *Martingale methods in financial modelling*, Springer, 2006.

[34] M. A. Rami and X. Y. Zhou, Linear matrix inequalities, Riccatie equations, and indefinite stochastic linear quadratic controls, *IEEE Trans. Automatic Control*, 45 (2000), 1131-1143.

[35] M. A. Rami, X. Chen, J. B. Moore, and X. Y. Zhou, Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls, *IEEE Trans. on Automatic Control*, 46 (2001), 428-440.

[36] F. D. Rouah, *The Heston model and its extensions in Matlab and C*, John Wiley & Sons, 2013.

[37] F. D. Rouah, *The Heston model and its extensions in VBA*, John Wiley & Sons, 2015.

[38] S. E. Shreve, *Stochastic calculus for finance II: continuous-time models*, Springer, 2004.

[39] C. F. Wang, H. Chang, and Z. M. Fang, Optimal portfolio and consumption rule with a CIR model under HARA utility, *Journal of the Operations Research Society of China*, 6 (2018), 107-137.

[40] J. Yong, Some estimates on exponentials of solutions to stochastic differential equations, *Journal of Applied Mathematics and Stochastic Analysis*, 4 (2004), 287-316.

[41] J. Yong and X. Y. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*, Springer, 1999.