Infinite-horizon optimal regulator for linear stochastic systems with state-delay

Nuha Alasmi[1,2] and Bujar Gashi[1]

***Abstract*— We consider the optimal linear-quadratic regulator problem on an *infinite-horizon* for linear stochastic control systems with discrete and distributed state-delay. The mean-square stability of such systems is first considered, and sufficient conditions for this system property to hold are derived. An explicit closed-form solution to the infinite-horizon linear-quadratic regulator problem is then derived, which turns out to be in a linear feedback form with respect to the system state, discrete state-delay, and distributed state-delay. The coefficients of such a control law depend of the solution to a matrix valued system of algebraic equations and inequalities, which generalise the Riccati algebraic equation.**

I. Introduction

Let $(\Omega, \mathscr{F}, (\mathscr{F}(t), t \geq 0), \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathscr{F}(t)$ is the augmentation of $\sigma\{W(s): 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$. Consider the following linear stochastic control system with discrete and distributed *state-delay* (for $t \geq 0$):

$$\begin{cases} dx(t) = [Ax(t) + Bx_h(t) + Cy(t) + Du(t)]dt \\ \quad +[A_1x(t) + B_1x_h(t) + C_1y(t) + D_1u(t)]dW(t), \\ x_h(t) := x(t-h), \\ y(t) := \int_{t-h}^{t} x(\theta)d\theta, \\ x(s) = \eta(s), \quad s \in [-h, 0], \quad \text{is given.} \end{cases} \tag{1}$$

Here $0 \leq h \in \mathbb{R}$; the constant coefficients $A, B, C, A_1, B_1, C_1 \in \mathbb{R}^{n\times n}$, and $D, D_1 \in \mathbb{R}^{n\times m}$, are given; the n-dimensional initial value η is assumed to be a continuous function on the interval $[-h, 0]$, i.e. $\eta(\cdot) \in C([-h, 0]; \mathbb{R}^n)$; and the adapted m-dimensional control process u is such that equation (1) has a unique strong solution for the n-dimensional system state x (see, for example, [14] for some sufficient conditions that ensure this). The processes x_h and y represent the discrete and distributed state-delay, respectively. Further consider the finite-horizon quadratic cost functional:

$$J_T(u(\cdot)) := \mathbb{E}\left\{\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt\right\}, \tag{2}$$

for some given constant symmetric coefficients $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$. The corresponding optimal linear-quadratic regulator problem is the following control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathscr{A}_T} J_T(u(\cdot)) \\ s.t. \quad (1) \end{cases}, \tag{3}$$

where $\mathscr{A}_T$ is some suitable set of admissible controls. One of the recent papers to give an explicit closed-form solution to problems of this type is [13]. The solution turns out to be in a feedback form with respect to the system state, the discrete state-delay x_h, and the distributed state-delay y. The coefficients of such a control law are determined by the solution to certain system of coupled Riccati and partial differential equations, the solvability of which is assumed (see [13] for details). This is an interesting result as not only it gives the solution in an explicit feedback closed-form, which is *rare* in optimal control, but it does so for the important class of systems with state-delay, which, as is well-known, appear in many applications and have been studied extensively for decades. The solution to (3) in an *open-loop* form can be obtained from the result of [8]. In [1] and [5], the problem (3) is generalised by permitting a *random* time-horizon, rather that a fixed one, whereas in [3] the more general case of coefficients with Markovian switching is considered. In [2], multiple state-delays are permitted, whereas [4] gives an application to the optimal investment problem of benchmark tracking. For some other related results, see, for example, [7], [10], [11], [12], [19], [20], [21].

In this paper, we consider an *infinite-horizon* version of problem (3). Thus, instead of the finite-horizon cost functional J_T, we consider its infinite-horizon analogue, defined as:

$$J(u(\cdot)) := \mathbb{E}\left\{\int_0^\infty \left[x'(t)Qx(t) + u'(t)Ru(t)\right]dt\right\}. \tag{4}$$

The corresponding infinite-horizon optimal regulator problem to be considered is:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathscr{A}} J(u(\cdot)) \\ s.t. \quad (1) \end{cases}, \tag{5}$$

where $\mathscr{A}$ is the admissible set of controls to be defined next. In order to ensure the finiteness of cost functional J, and thus the well-posedness of problem (5), we impose a certain *stability* requirement on the control processes. Namely, the set $\mathscr{A}$ consists of all $\mathbb{R}^m$-valued adapted stochastic processes u under which equation (1) has a unique strong solution, and there exist constants $\alpha > 0$ and $\gamma >$ such that $\mathbb{E}[x'(t)x(t) +$

[1]Department of Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK. N.Alasmi@liverpool.ac.uk and Bujar.Gashi@liverpool.ac.uk
[2]Department of Mathematics, The University of Jeddah, KSA.

$y'(t)y(t)] \leq \alpha ||\eta||^2 e^{-\gamma t}$ for all $\eta(\cdot) \in C([-h,0];\mathbb{R}^n)$, where $||\cdot||$ denotes the L^2-norm. This requirement in particular ensure the asymptotic mean-square stability of the system.

In order to obtain the solution to problem (5), we begin in §II by considering the mean-square asymptotic stability of system (1). Some sufficient condition for this system property to hold are derived, and an appropriate notion of stability is introduced, which is particularly suitable for the infinite-horizon optimal regulator problem. In §III, we derive the solution to problem (5) in an explicit closed-form. Similarly to the finite-horizon case, the optimal control law is in a feedback form with respect to the system state x, the discrete state-delay x_h, and the distributed state-delay y. But instead of the system of coupled Riccati and partial differential equations that appear in finite-horizon, we now have a system of matrix-valued algebraic equations and inequalities. Finally, with regards to notation, we omit the argument t whenever convenient for simplicity; I and 0 will denote the identity and zero matrices, respectively, of appropriate dimensions (which will be clear from the context); and A' denotes the transpose of a matrix A.

II. Mean-square exponential stability

In this section we consider the stability problem for the uncontrolled system (1), the equation of which is:

$$\begin{cases} dx(t) = [Ax(t) + Bx_h(t) + Cy(t)]dt \\ +[A_1x(t) + B_1x_h(t) + C_1y(t)]dW(t), \\ x(s) = \eta(s), \quad s \in [-h,0], \quad \text{is given.} \end{cases} \tag{6}$$

Our first result gives sufficient conditions for the mean-square exponential stability of (6). This is a certain generalisation of Theorem 3.1 of [15], which gives sufficient conditions for the stability of a certain class of linear stochastic systems with state-delay. In [15], the system coefficients are permitted to be modulated by a continuous-time Markov chain, however, only discrete delays are considered, and thus when speciated to the case without Markovian switching, such a result does not cover system (6). In order the generalise such a result, we introduce the following processes and matrices:

$$z(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad z_h(t) := \begin{bmatrix} x(t-h) \\ y(t-h) \end{bmatrix}, \quad t \geq 0;$$

$$\mu(s) := \begin{bmatrix} \eta(s) \\ \int_{-h}^{0} \eta(\theta) d\theta \end{bmatrix}, \quad s \in [-h,0],$$

$$||\mu||^2 := \int_{-h}^{0} \mu'(s)\mu(s)ds;$$

$$F := \begin{bmatrix} A & C \\ I & 0 \end{bmatrix}, \quad G := \begin{bmatrix} B & 0 \\ -I & 0 \end{bmatrix},$$

$$F_1 := \begin{bmatrix} A_1 & C \\ 0 & 0 \end{bmatrix}, \quad G := \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix};$$

$$H_1(K) := KF + F'K + I + F_1'KF_1,$$

$$H_2(K) := KG + F_1'KG_1,$$

$$H_3(K) := -I + G_1'KG_1,$$

$$H(K) := \begin{bmatrix} H_1 & H_2 \\ H_2' & H_3 \end{bmatrix}, \quad \text{where } K \in \mathbb{R}^{2n\times 2n}.$$

.

Theorem 2.1: If there exists a symmetric positive definite matrix $K \in \mathbb{R}^{2n\times 2n}$ such that $H(K) < 0$, then there exist $\alpha > 0$ and $\gamma > 0$ such that:

$$\mathbb{E}\left[z'(t)z(t)\right] \leq \alpha e^{-\gamma t} ||\mu||^2 \tag{7}$$

for any $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$, and

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[x'(t)x(t)\right] \leq -\gamma. \tag{8}$$

Proof. As $dy = (x - x_h)dt$, the equation of z is:

$$\begin{cases} dz = (Fz + Gz_h)dt + (F_1z + G_1z_h)dW(t), \quad t \geq 0, \\ z(s) = \mu(s), \quad s \in [-h,0]. \end{cases} \tag{9}$$

Since (9) is a linear stochastic system with only discrete state-delay, Theorem 3.1 of [15] applies, and thus inequality (7) follows from the last inequality in the proof of Theorem 3.1 of [15]). Due to (7), $\mathbb{E}[x'(t)x(t)] \leq \mathbb{E}[z'(t)z(t)] \leq \alpha e^{-\gamma t}||\mu||^2$, from which (8) follows. □

Our second result, which is based on Theorem 2.1, gives a form of asymptotic stability appropriate for the infinite-horizon optimal regulator problem to be solved in the next section. Consider the process S defined as:

$$\begin{aligned} S(t) \quad &:= \quad x'(t)L_1x(t) + 2x'(t)L_2y(t) + y'(t)L_3y(t) \\ &+ \quad \int_{t-h}^{t} x'(\theta)L_4x(\theta)d\theta, \end{aligned}$$

where $L_1, L_2, L_3, L_4 \in \mathbb{R}^{n\times n}$, with L_1, L_3, L_4 assumed symmetric.

Theorem 2.2: If $L_1 > 0$, $L_3 - L_2'L_1^{-1}L_2 \geq 0$, $L_4 \geq 0$, and there exists a symmetric positive definite matrix $K \in \mathbb{R}^{2n\times 2n}$ such that $H(K) < 0$, then

$$\lim_{t\to\infty} \mathbb{E}[S(t)] = 0 \tag{10}$$

for any $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$.

Proof. Let the matrix L be defined as:

$$L := \begin{bmatrix} L_1 & L_2 \\ L_2' & L_3 \end{bmatrix}.$$

By Schur's lemma (see, for example, Lemma 3 in [16]), and the assumptions on L_1, L_2, L_3, L_4, it follows that $L \geq 0$. It further follows from Theorem 2.1 and its proof that there exist $\alpha > 0$ and $\gamma > 0$ such that $\mathbb{E}[z'(t)z(t)] \leq \alpha e^{-\gamma t}||\mu||^2$

and $\mathbb{E}[x'(t)x(t)] \le \alpha e^{-\gamma t}||\mu||^2$ for any $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$. Thus:

$$0 \le \mathbb{E}[z'(t)Lz(t)] \le \lambda_L \mathbb{E}[z'(t)z(t)] \le \lambda_L \alpha e^{-\gamma t}||\mu||^2, \quad (11)$$

where $\lambda_L \ge 0$ is the largest eigenvalue of L (the second inequality in (11) being due to Rayleigh quotient). As the last term of (11) converges to zero, and $\mathbb{E}[z'(t)Lz(t)]$ is bounded from below by zero, it follows that:

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}[z'(t)Lz(t)] &= \lim_{t\to\infty}\mathbb{E}[x'(t)L_1x(t)+2x'(t)L_2y(t)\\ &+ y'(t)L_3y(t)] = 0, \quad (12)\end{aligned}$$

for any $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$. If $\lambda_{L_4} \ge 0$ denotes the largest eigenvalue of L_4, then:

$$\begin{aligned}0 &\le \mathbb{E}\left[\int_{t-h}^{t} x'(\theta)L_4x(\theta)d\theta\right] \le \mathbb{E}\left[\int_{t-h}^{t} \lambda_{L_4}x'(\theta)x(\theta)d\theta\right]\\ &\le \lambda_{L_4}\alpha||\mu||^2\int_{t-h}^{t} e^{-\gamma\theta}d\theta\\ &= \lambda_{L_4}\alpha||\mu||^2\gamma^{-1}\left[e^{-\gamma(t-h)}-e^{-\gamma t}\right]. \quad (13)\end{aligned}$$

As the last term in (13) converges to zero as $t\to\infty$, it follows from (13) that:

$$\lim_{t\to\infty}\mathbb{E}\left[\int_{t-h}^{t} x'(\theta)L_4x(\theta)d\theta\right] = 0 \quad (14)$$

for any $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$. The limit (10) now follows from (12) and (14). □

III. Solution to the optimal regulator problem

In this section we give the solution to infinite-horizon regulator problem (5) in an explicit closed-form. In order to state such a solution and the assumptions under which it holds, we introduce the following set of matrix-valued algebraic equations and inequalities:

$$\begin{cases} P_1A+A'P_1+A_1'P_1A_1+P_2+P_2'+P_4\\ +Q-E_1\bar{R}^{-1}E_1'=0,\\ P_1B+A_1'P_1B_1-P_2-E_1\bar{R}^{-1}E_2'=0,\\ P_1C+A_1'P_1C_1+A'P_2+P_3-E_1\bar{R}^{-1}E_3'=0,\\ B_1'P_1B_1-P_4-E_2\bar{R}^{-1}E_2'=0,\\ B_1'P_1C_1+B'P_2-P_3-E_2\bar{R}^{-1}E_3'=0,\\ C_1'P_1C_1+C'P_2+P_2'C-E_3\bar{E}^{-1}E_3=0,\\ P_1>0,\\ P_3-P_2'P_1^{-1}P_2\ge 0\\ P_4\ge 0,\\ \bar{R}>0.\end{cases} \quad (15)$$

Here $P_1,P_2,P_3,P_4 \in \mathbb{R}^{n\times n}$, with P_1, P_3, P_4 assumed symmetric, and

$$\bar{R} := R+D_1'P_1D, \qquad E_1 := P_1D+A_1'P_1D_1,$$

$$E_2 := B_1'P_1D_1, \qquad E_3 := C_1'P_1D_1+P_2'D.$$

We further introduce the following matrix-valued functions of P_1 and P_2:

$$\tilde{A} := A-D\bar{R}^{-1}E_1', \quad \tilde{A}_1 := A_1-D_1\bar{R}^{-1}E_1'$$

$$\tilde{B} := B-D\bar{R}^{-1}E_2', \quad \tilde{B}_1 := B_1-D_1\bar{R}^{-1}E_2',$$

$$\tilde{C} := C-D\bar{R}^{-1}E_3', \quad \tilde{C}_1 := C_1-D_1\bar{R}^{-1}E_3',$$

$$\tilde{F} := \begin{bmatrix}\tilde{A} & \tilde{C}\\ I & 0\end{bmatrix}, \quad \tilde{G} := \begin{bmatrix}\tilde{B} & 0\\ -I & 0\end{bmatrix},$$

$$\tilde{F}_1 := \begin{bmatrix}\tilde{A}_1 & \tilde{C}\\ 0 & 0\end{bmatrix}, \quad \tilde{G} := \begin{bmatrix}\tilde{B} & 0\\ 0 & 0\end{bmatrix};$$

Finally, we introduce the following functions of P_1, P_2, and of a matrix $\tilde{K} \in \mathbb{R}^{2n\times 2n}$:

$$\tilde{H}_1(\tilde{K}) := \tilde{K}\tilde{F}+\tilde{F}'\tilde{K}+I+\tilde{F}_1'\tilde{K}\tilde{F}_1,$$

$$\tilde{H}_2(\tilde{K}) := \tilde{K}\tilde{G}+\tilde{F}_1'\tilde{K}\tilde{G}_1,$$

$$\tilde{H}_3(\tilde{K}) := -I+\tilde{G}_1'\tilde{K}\tilde{G}_1,$$

$$\tilde{H}(\tilde{K}) := \begin{bmatrix}\tilde{H}_1 & \tilde{H}_2\\ \tilde{H}_2' & \tilde{H}_3\end{bmatrix}.$$

Assumption 1. *There exist a unique solution (P_1,P_2,P_3,P_4) to (15) such that $\tilde{H}(\tilde{K}) < 0$ for some positive definite symmetric matrix $\tilde{K} \in \mathbb{R}^{2n\times 2n}$.*

The system of matrix-valued equations and inequalities (15) is a generalisation to the algebraic Riccati equation of stochastic linear-quadratic control for systems without state-delay (see, for example, [16], [17]). At the end of this section, in Example 3.1, we derive some sufficient conditions for the unique solvability of (15) in one particular case.

Theorem 3.1: There exists a unique solution u^* to the infinite-horizon optimal regulator problem (5) given as:

$$u^*(t) := -\bar{R}^{-1}\left[E_1'x(t)+E_2'x(t-h)+E_3'\int_{t-h}^{t}x(\theta)d\theta\right], \quad t\ge 0.$$

The corresponding optimal cost functional is:

$$J(u^*(\cdot)) = x'(0)P_1x(0) + 2x'(0)P_2\int_{-h}^{0}\eta(\theta)d\theta + \int_{-h}^{0}\int_{-h}^{0}\eta'(\theta)P_3\eta(s)d\theta ds + \int_{-h}^{0}\eta'(\theta)P_4\eta(\theta)d\theta.$$

Moreover, under the control u^*, there exists $\gamma > 0$ such that:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[x'(t)x(t)\right] \leq -\gamma. \tag{16}$$

for all $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$.

Proof. The process V is defined as (for all $t \geq 0$):

$$V(t) := x'P_1x + 2x'P_2y + y'P_3y + \int_{t-h}^{t}x'(\theta)P_4x(\theta)d\theta.$$

By Itô's formula, the differentials of terms appearing in V are:

$$d(x'P_1x) = [2x'P_1(Ax+Bx_h+Cy+Du)] + (A_1x+B_1x_h + C_1y+D_1u)'P_1(A_1x+B_1x_h+C_1y+D_1u)]dt + 2x'P_1(A_1x+B_1x_h+C_1y+D_1u)dW,$$

$$d(2x'P_2y) = [2(Ax+By_h+Cy+Du)'P_2y + 2x'P_2(x-x_h)]dt + 2(A_1x+B_1x_h+C_1y+D_1u)'P_2ydW,$$

$$d(y'P_3y) = 2y'P_3(x-x_h)dt,$$

$$d\left(\int_{t-h}^{t}x'(\theta)P_4x(\theta)d\theta\right) = (x'P_4x - x_h'P_4x_h)dt.$$

By integrating the above differentials from 0 to t, and then taking their expectation, we can write $\mathbb{E}[V(t)]$ for any $u(\cdot) \in \mathscr{A}$ as:

$$\mathbb{E}[V(t)] = V(0) + \mathbb{E}\left\{\int_0^t [x'(P_1A + A'P_1 + A_1'P_1A_1 + P_2 + P_2' + P_4)x + 2x'(P_1B + A_1'P_1B_1 - P_2)x_h + 2x'(P_1C + A_1'P_1C_1 + A'P_2 + P_3)y + 2x'(P_1D + A_1'P_1D_1)u + x_h'(B_1'P_1B_1 - P_4)x_h + 2x_h'(B_1'P_1C_1 + B'P_2 - P_3)y \; 2x_h'B_1'P_1D_1u + y'(C_1'P_1C_1 + C'P_2)y + 2y'(C_1'P_1D_1 + P_2'D)u + u'D_1P_1D_1u]dt\right\}. \tag{17}$$

As for any $u(\cdot) \in \mathscr{A}$, there exist $\alpha > 0$ and $\gamma > 0$ such that $\mathbb{E}[z'(t)z(t)] \leq \alpha e^{-\gamma t}||\mu||^2$, for all $\eta(\cdot) \in C([-h,0],\mathbb{R}^n)$, it follows from Theorem 2.2 and its proof that:

$$\lim_{t\to\infty}\mathbb{E}[V(t)] = 0.$$

Thus, due to (17), for any $u(\cdot) \in \mathscr{A}$ the following holds:

$$0 = V(0) + \mathbb{E}\left\{\int_0^\infty [x'(P_1A + A'P_1 + A_1'P_1A_1 + P_2 + P_2' + P_4)x + 2x'(P_1B + A_1'P_1B_1 - P_2)x_h + 2x'(P_1C + A_1'P_1C_1 + A'P_2 + P_3)y + 2x'(P_1D + A_1'P_1D_1)u + x_h'(B_1'P_1B_1 - P_4)x_h + 2x_h'(B_1'P_1C_1 + B'P_2 - P_3)y \; 2x_h'B_1'P_1D_1u + y'(C_1'P_1C_1 + C'P_2)y + 2y'(C_1'P_1D_1 + P_2'D)u + u'D_1P_1D_1u]dt\right\}. \tag{18}$$

By adding the right-hand side of (18) to $J(u(\cdot))$, we can write this cost functional as:

$$J(u(\cdot)) = V(0) + \mathbb{E}\left\{\int_0^\infty [x'(P_1A + A'P_1 + A_1'P_1A_1 + P_2 + P_2' + P_4 + Q)x + 2x'(P_1B + A_1'P_1B_1 - P_2)x_h + 2x'(P_1C + A_1'P_1C_1 + A'P_2 + P_3)y + 2x'(P_1D + A_1'P_1D_1)u + x_h'(B_1'P_1B_1 - P_4)x_h + 2x_h'(B_1'P_1C_1 + B'P_2 - P_3)y \; 2x_h'B_1'P_1D_1u + y'(C_1'P_1C_1 + C'P_2)y + 2y'(C_1'P_1D_1 + P_2'D)u + u'(R + D_1P_1D_1)u]dt\right\}. \tag{19}$$

The terms in the integrand of (19) that depend explicitly of the control u are:

$$u'(R + D_1'P_1D_1)u + 2x'(P_1D + A_1'P_1D_1)u + 2x_h'B_1'P_1D_1u + 2y'(C_1'P_1D_1 + P_2'D)u$$
$$= u'\bar{R}u + 2(x'E_1 + x_h'E_2 + y'E_3)u$$
$$= (u-u^*)'\bar{R}(u-u^*) - (x'E_1 + x_h'E_2 + yE_3)\bar{R}^{-1}(E_1'x_1 + E_2'x_h + E_3'y),$$

where the last equality is obtained by completion of squares.

We can now write (19) as:

$$J(u(\cdot)) = V(0)+\mathbb{E}\left\{\int_0^\infty (u-u^*)'\bar{R}(u-u^*)dt\right\} \geq V(0), \quad (20)$$

which holds for all $u(\cdot)\in\mathscr{A}$. The lower bound (20) is achieved if and only if $u(t)=u^*(t)$ for a.e. $t\geq 0$ a.s.. Thus, if $u^*(\cdot)\in\mathscr{A}$, then it is the claimed unique solution to the infinite-horizon optimal regulator problem (5). Under the control u^*, equation (1) can be written as:

$$dx=(\tilde{A}x+\tilde{B}x_h+\tilde{C}y)dt+(\tilde{A}_1x+\tilde{B}_1x_h+\tilde{C}_1y)dW.$$

As this system satisfies the requirement of Theorem 2.1 (due to Assumption 1), it follows by that theorem that there exist $\alpha>0$ and $\gamma>0$ such that $\mathbb{E}[z'(t)z(t)]\leq\alpha e^{-\gamma t}||\mu||^2$, for all $\eta(\cdot)\in C([-h,0],\mathbb{R}^n)$, and thus proving $u^*(\cdot)\in\mathscr{A}$. The mean-square asymptotic stability of the system (16) also follows from Theorem 2.1. □

Example 3.1: Here we give a simple example illustrating the solvability of the matrix-valued system of algebraic equations and inequalities (15). Thus, let $B=0$, $C=0$, $A_1=0$, $C_1=0$, $D_1=0$, $R>0$, and $Q\geq 0$, which corresponds to the following linear stochastic control system with discrete state-delay:

$$dx(t)=[Ax(t)+Du(t)]dt+B_1x(t-h)dW(t).$$

In this case, $\bar{R}=R$, $E_1=P_1D$, $E_2=0$, $E_3=P_2'D$, and the equations in (15) become:

$$P_1A+A'P_1+P_2+P_2'+P_4+Q-P_1DR^{-1}D'P_1=0, \quad (21)$$

$$-P_2=0, \quad (22)$$

$$A'P_2+P_3-P_1DR^{-1}D'P_2=0, \quad (23)$$

$$B_1'P_1B_1-P_4=0, \quad (24)$$

$$B'P_2-P_3=0, \quad (25)$$

$$-P_2'DR^{-1}P_2=0. \quad (26)$$

It follows from (22) that $P_2=0$. This implies that (26) holds, and that $P_3=0$ is the solution to (25). Thus, (23) also holds. From (24) it follows that $P_4=B_1'P_1B_1$. The equation (21) now becomes:

$$P_1A+A'P_1+B_1'P_1B_1+Q-P_1DR^{-1}D'P_1=0, \quad (27)$$

This is an example of the algebraic Riccati equation of stochastic linear-quadratic control. By Theorem 4.1 of [18], if (A,D) is stabilizable, $(Q^{1/2},A)$ is observable, and

$$\inf_{X\in\mathbb{R}^{m\times n}}\left|\int_0^\infty e^{t(A-DX)'}B_1'B_1e^{t(A-DX)}dt\right|<1,$$

with $|\cdot|$ denoting the Euclidean matrix norm, there exists a unique positive definite solution P_1 to equation (27). This implies that P_4 is also unique. It is clear that all inequalities of (15) hold. □

IV. CONCLUSIONS

We have considered the infinite-horizon optimal regulator problem for a class of linear stochastic systems with discrete and distributed state-delays. Such delays appear in both the drift and diffusion terms of the state equation. After considering the mean-square asymptotic stability, we derive an explicit closed-form solution to this problem in a feedback form. A certain system of matrix-valued algebraic equations and inequalities appear in this solution, which represent a generalisation to the Riccati algebraic equation of stochastic linear-quadratic control. Although the finite-horizon problem has been studied extensively, there appears to be considerably less results for the infinite-horizon problem. Our approach to obtaining the solution has a potential for applications to other settings, such as systems with Markovian switching coefficients, with multiple delays, with input delays, as well as state-delay versions of nonlinear systems considered in [6], [9].

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