Differential Exponential Stability of Nonlinear Time-Invariant Systems

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***Abstract*— Motivated by the limitations of standard eigenvalue approach to study the stability of nonlinear systems, we propose a dynamical spectral decomposition based on a novel operator eigenproblem, generalizing the classical Jacobian-based analysis for equilibrium points. The proposed framework bridges operator-theoretic and spectral perspectives on stability, offering new tools for the study of a broad class of nonlinear dynamical systems. For completeness, we compare our operator eigenproblem with the standard eigenvalue problem and provide a generalization of the eigenproblem framework from which we derive two conditions for differential exponential stability. To illustrate the approach, we analyze the differential stability properties of a nonlinear system.**

I. INTRODUCTION

Stability analysis is a central theme in systems and control theory, supporting both qualitative understanding of dynamical behavior and the design of controllers and observers. For nonlinear autonomous systems, stability of an equilibrium is most commonly certified by constructing a Lyapunov function whose derivative is strictly negative away from the equilibrium [20].

For linear time-invariant (LTI) systems, asymptotic stability admits a complete spectral characterization: it is determined by the negativeness of the real parts of the eigenvalues of the state matrix, due to the direct link between the spectrum and the behavior of solution [1], [25]. In contrast, for time-dependent dynamics this “instantaneous” picture breaks down. Pointwise spectral information is in general neither necessary nor sufficient to infer stability. In particular, a linear time-varying (LTV) system may be unstable even when all instantaneous eigenvalues of the system matrix have strictly negative real parts at every time instant [1], [3], [6], [9], [20], [24], [25], [29], [30].

This limitation reflects the fact that stability of LTV systems is governed by the effect of the time-varying nature of the state matrix eigenstructure, rather than by the state matrix eigenvalues at fixed time instants. For example, rotating or rapidly changing eigenspaces can hide an underlying divergence, as shown in [26], despite apparently “stable” instantaneous spectra. Accordingly, LTV stability criteria are typically expressed in terms of the state-transition matrix, which encodes the global-in-time evolution of the dynamics [8], [13], [32].

To recover a spectral viewpoint in the time-varying setting, several notions extending eigenvalues/eigenvectors have been proposed. Early work introduced *dynamical eigenvectors* for LTV systems [31]; later developments formalized nonlinear eigenvalues for nonlinear systems [14] and established nonlinear spectral theories aimed at stability analysis [17]–[19]. Further developments and applications appear in [22]. Related ideas have also been used to characterize analytic features of solutions of LTV systems [11]. On the analytical side, recent efforts have connected nonlinear eigenvalues/eigenvectors and operator-theoretic formulations. In particular, Koopman operator eigenfunctions have been used as stability tools in [21].

In this paper, we advance the stability analysis from a differential perspective. The central contribution of this paper is the introduction of a novel operator-theoretic formulation whose spectral structure allows us also to explicitly construct the state-transition matrix of the variational dynamics. Furthermore, this spectral decomposition generalizes the classical eigenvalue-based approach used in the analysis of LTI systems, providing a spectral decomposition applicable to the nonlinear setting. We leverage the obtained eigenstructure to derive differential exponential stability criteria, without requiring integrability of the eigenfunctions. Furthermore, we also provide the definition of a weaker change of coordinates condition, which allows us to derive two differential exponential stability criteria.

The remainder of the paper is organized as follows. Section II reviews linear operators and their eigenstructure, and recalls Kawano’s notions of nonlinear eigenvectors and eigenvalues for autonomous systems. Section III then introduces the framework of nonlinear time-varying systems, their associated variational dynamics, and the notion of differential exponential stability used throughout this paper. Section IV, introduces a dynamical spectral decomposition based on the eigenproblem of a newly defined operator and provides a comparison between the dynamical and classical Jordan spectral decomposition and a generalization of the approach. Section V, then employs this generalization to derive two differential exponential stability criteria for nonlinear systems. Finally, in Section VI, we analyze the differential stability properties of a nonlinear system, before the conclusions in Section VII.

Notation: The set of real (complex) numbers is denoted by $\mathbb{R}$ ($\mathbb{C}$), and $\mathbb{N}$ denotes the set of natural numbers. The real part of a complex number $x \in \mathbb{C}$ is denoted as $\Re(x) \in \mathbb{R}$. Given $n \in \mathbb{N}$, vectors are assumed to belong to $\mathbb{R}^n$ and matrices to $\mathbb{R}^{n\times n}$. An $n \times n$ real matrix with zero entries is denoted as $\mathbf{0}_{n\times n}$. The symbol $\|\cdot\|$ is used as the standard

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Euclidean norm for vectors and the induced 2-norm for matrices, unless stated otherwise.

$\mathcal{C}^k(\mathcal{X})$ denotes the class of continuously k times differentiable functions on $\mathcal{X}$. Given a differentiable function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the Jacobian matrix with respect to the state variable is denoted by $\frac{\partial f(x)}{\partial x}$.

For a matrix $M \in \mathbb{R}^{n\times n}$, $\sigma(M)$ denotes its spectrum, $\lambda_i(M)$ its i-th eigenvalue, and $\sigma_i(M)$ its i-th singular value, defined as $\sigma_i(M) = \sqrt{\lambda_i(M^\top M)}$. The minimum and maximum singular values are denoted by $\sigma_{\min}(M)$ and $\sigma_{\max}(M)$, respectively, and the condition number of M is $\mu(M) = \dfrac{\sigma_{\max}(M)}{\sigma_{\min}(M)}$. The 2-norm for matrices comes then directly as $\|M\| = \sigma_{\max}(M)$ and $\|M^{-1}\| = \sigma_{\min}(M)^{-1}$.

A matrix P is said to be symmetric if $P = P^\top$. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $P > 0$, if $x^\top P x > 0$, for all $x \in \mathbb{R}^n \setminus \{0\}$.

II. PRELIMINARIES

We begin by recalling the eigenproblem in the general settings of operators and the nonlinear eigenstructure introduced in [18].

A. Operators, eigenvalues, and eigenfunctions

In this subsection, we present the extension of the classical notions of eigenvalues and eigenvectors from linear algebra to a more general functional analysis setting. Here, operators act on function spaces rather than finite-dimensional vector spaces, and eigenvalues may themselves be scalar-valued functions of the independent variables. This framework is fundamental for analyzing systems whose spectral properties depend on the state $x \in \mathcal{X} \subseteq \mathbb{R}^m$.

Definition II-A.1 (Operator)**.** Let $\mathcal{X} \subseteq \mathbb{R}^m$ (or $\mathbb{C}^m$) and let $\mathcal{F}$ be a suitable space of functions $f : \mathcal{X} \to \mathbb{R}^n$ (or $\mathbb{C}^n$). An *operator* is a mapping $\mathscr{O} : \mathcal{F} \to \mathcal{F}, \quad f \mapsto \mathscr{O}[f]$, which assigns to each function $f \in \mathcal{F}$ a new function $\mathscr{O}[f] \in \mathcal{F}$ defined on $\mathcal{X}$. ⌟

Definition II-A.2 (Eigenvalue and eigenfunction)**.** Let $\mathscr{O} : \mathcal{F} \to \mathcal{F}$ be an operator. A nonzero function $T_j \in \mathcal{F}$ is called an *eigenfunction* of $\mathscr{O}$ if there exists a map $\lambda_j : \mathcal{X} \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$\mathscr{O}[T_j](x) = \lambda_j(x)\, T_j(x), \quad \forall x \in \mathcal{X}.$$

The map λ_j is called the *eigenvalue function* associated with the eigenfunction T_j. ⌟

Remark II.1 By *pointwise linear independence* of the eigenfunctions of an operator we mean that the distribution of

$$\big(T_1, \dots, T_n\big)$$

spans $\mathbb{R}^n$ at each point of $\mathcal{X}$. Equivalently, for column-vector eigenfunctions $T_j : \mathcal{X} \to \mathbb{C}^n$, $j = 1, \dots, n$, the matrix

$$T(x) := \big[\, T_1(x) \,\big|\, \cdots \,\big|\, T_n(x) \big] \in \mathbb{C}^{n\times n}$$

is invertible at every point of the domain, i.e.,

$$\det\big(T(x)\big) \neq 0, \quad \forall x \in \mathcal{X}.$$ ⌟

B. Nonlinear eigenvectors and eigenvalues

We now recall the nonlinear system eigenstructure framework developed in [18]. Therein, the authors analyze the conditions under which a nonlinear system $\dot{x} = f(x)$ admits a diffeomorphic change of coordinates that diagonalized its dynamics. Specifically, a system $\dot{x} = f(x)$ is said to be diagonalizable on a compact set $\mathcal{X}_d \subseteq \mathcal{X}$ if there exists a diffeomorphism $\phi : \mathcal{X}_d \to \mathbb{C}^n$ such that, in the new coordinates $\mathrm{z} = \phi(x)$, the dynamics simplifies to the decoupled form $\dot{\mathrm{z}}_i = g_i(\mathrm{z}_i)$, $i = 1, \dots, n$, where each state $\mathrm{z}_i : \mathbb{R} \to \mathbb{C}$ and its dynamics $g_i : \mathbb{C} \to \mathbb{C}$ is analytic. In this context, the nonlinear eigenvalue and eigenvectors of the system are defined as follows.

Definition II-B.1 (Def. 2 in [18])**.** Let $\lambda : \mathcal{X}_d \to \mathbb{C}$ and $v : \mathcal{X}_d \to \mathbb{C}^n$ be complex analytic functions, such that $v(x^\star) \neq 0$ at any equilibrium point $x = x^\star$. They are called, respectively, a *nonlinear eigenvalue* and *nonlinear left eigenvector* of system $\dot{x} = f(x)$ if, for all $x \in \mathcal{X}_d$,

$$v^\top(x)\frac{\partial f(x)}{\partial x} + \left(\frac{\partial v(x)}{\partial x} f(x)\right)^\top = \lambda(x) v^\top(x). \quad (1)$$ ⌟

The authors further provide necessary and sufficient conditions for the diagonalizability of the nonlinear system, summarized below.

Theorem II-B.1 (Th.1 in [18])**.** *Let $\lambda_i : \mathcal{X}_d \to \mathbb{C}$ and $v_i : \mathcal{X}_d \to \mathbb{C}^n$, denote the nonlinear eigenvalues and their associated left eigenvectors of the system $\dot{x} = f(x)$. The system is diagonalizable on $\mathcal{X}_d \subset \mathbb{R}^n$ if and only if*

a) the distribution of $(v_1(x), \dots, v_n(x))$ spans $\mathbb{R}^n$ for every $x \in \mathcal{X}_d$;

b) there exist complex analytic functions $\psi_i : \mathcal{X}_d \to \mathbb{C}^n$ such that $v_i^\top(x)dx = d\psi$ on $\mathcal{X}_d$, $i = 1, \dots, n$. ⌟

The first condition establishes pointwise linear independence of the nonlinear eigenvectors, ensuring that they form a basis of the tangent space at each $x \in \mathcal{X}_d$. The second condition is an integrability requirement, which guarantees that the vector field, associated with each left eigenfunction, is locally exact and therefore admits a scalar potential function. This property enables the explicit construction of decoupling coordinates $\mathrm{z} = \phi(x)$ through analytic integration, allowing the change of coordinates defined in the *lifted* variational space to be consistently *pulled down* to the underlying state space.

III. THE DYNAMICAL SYSTEM CONTEXT

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be an arbitrary connected subset of $\mathbb{R}^n$, and let $f : \mathcal{X} \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map in (x). Consider a *forward complete*[1] dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad (2)$$

where $x : \mathbb{R} \to \mathcal{X} \subseteq \mathbb{R}^n$ and $t \in [t_0, \infty) \subseteq \mathbb{R}$. The map f is the system's vector field, and $x(t)$ denotes a local coordinate

[1] A system is called forward complete if, for every initial condition, the corresponding solution is defined for all forward times [4].

representation of a system's trajectory. Given system (2), we denote by $\varphi(x_0, t; t_0) \in \mathcal{X}$ the unique solution $x(t) = \varphi(x_0, t; t_0)$ starting at $x(t_0) = x_0 \in \mathcal{X}$ at time $t_0 \geq 0$. The map $\varphi : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ is called the system *flow* and has a relevant differentiability properties, i.e., if $f(x)$ is $\mathcal{C}^k$ in x, then $\varphi(x_0, t; t_0)$ is $\mathcal{C}^k$ with respect to x_0.

To study the sensitivity of trajectories with respect to perturbations in the initial conditions, we introduce the following smooth family of solutions. For a reference initial state $x_0 \in \mathcal{X}$, consider a family of parameterized trajectories $\chi : (-\delta_\chi, \delta_\chi) \times \mathbb{R}_{\geq 0} \to \mathcal{X}$ such that, for each $\varepsilon \in (-\delta_\chi, \delta_\chi)$, with $\delta_\chi \in \mathbb{R}_{>0}$ sufficiently small, the curve $t \mapsto \chi(\varepsilon, t)$ satisfies $\dot{\chi}(\varepsilon, t) = f(\chi(\varepsilon, t)), \quad \chi(\varepsilon, t_0) = x_0 + \varepsilon\nu$, where $\nu \in \mathbb{R}^n$ is an initial perturbation direction. The nominal trajectory is recovered for $\varepsilon = 0$, i.e., $\chi(0, t) = \varphi(x_0, t; t_0) = x(t)$. We define the *variational state* as $\delta x(t) := \lim_{\varepsilon \to 0} \frac{\partial \chi(\varepsilon, t)}{\partial \varepsilon}$

The evolution of $\delta x(t)$ is governed by a linear time-varying system known as the *variational dynamics*, which captures system (2) infinitesimal behavior around the nominal solution $\varphi(x_0, t; t_0)$, i.e.,

$$\dot{\delta x}(t) = \frac{\partial f}{\partial x}\Big(\varphi(x_0, t; t_0)\Big)\, \delta x(t), \quad \delta x(t_0) = \nu, \tag{3}$$

where $\nu \in \mathbb{R}^n$ denotes an admissible perturbation direction at time t_0. Classically, and directly from the definition of variational state, this linear time-varying and state-dependent system characterizes the evolution of infinitesimal deviations from the reference trajectory $\varphi(x_0, t; t_0)$ and plays a fundamental role in the stability and local sensitivity analysis of dynamical systems [2], [5], [16].

As any LTV system, the evolution of variational state is described by the *state transition matrix*, $\Phi : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ defined such that $\delta x(t) = \Phi(t, t_0)\delta x(t_0)$, for all $t, t_0 \in \mathbb{R}$.

Differential exponential stability: The variational system (3), associated with the dynamics (2), is regarded as a time-varying linear system whose stability properties depend simultaneously on the state trajectory $x(t)$ and on the time variable t. Furthermore, any stability property defined for this dynamics is interpreted as a *differential* stability property of the original nonlinear system (2).

Definition III-.1 (differential Regional Stability (dRS))**.** System (2) is said to be *differentially (regionally) stable* in a region $\overline{\mathcal{X}} \subseteq \mathcal{X}$ if, for any trajectory $\varphi(x_0, t; t_0)$ with $x_0 \in \overline{\mathcal{X}}$, the solution $\delta x(t)$ of the associated variational dynamics remains bounded for all forward time, i.e., for every $\varepsilon > 0$ there exists a $\delta = \delta_{\varepsilon, x_0, t_0} > 0$ such that, any initial infinitesimal displacement $\delta x(t_0) \in \mathbb{R}^n$ satisfying

$$\|\delta x(t_0)\| < \delta \implies \|\delta x(t)\| < \varepsilon, \ \forall t \geq t_0. \tag{4}$$

⌟

Definition III-.2 (differential Regional Asymptotic Stability (dRAS))**.** System (2) is said to be *differentially (regionally) asymptotically stable* in a region $\overline{\mathcal{X}} \subseteq \mathcal{X}$ if, for any initial condition $x_0 \in \overline{\mathcal{X}}$, it is differentially (regionally) stable and the solution $\delta x(t)$ of the variational system satisfies

$$\lim_{t \to \infty} \|\delta x(t)\| = 0, \quad \forall\, x_0 \in \overline{\mathcal{X}}.$$

⌟

Definition III-.3 (differential Regional Exponential Stability (dRES))**.** System (2) is *differentially (regionally) exponentially stable* in a region $\mathcal{X}$ if there exist real constants $\kappa = \kappa_{x_0, t_0} > 0$ and $\alpha = \alpha_{x_0, t_0} > 0$ such that, for all initial conditions $x_0 \in \mathcal{X}_0 \subseteq \mathcal{X}$ and $\delta x(t_0) \in \mathbb{R}^n$,

$$\|\delta x(t)\| \leq \kappa\, e^{-\alpha(t - t_0)} \|\delta x(t_0)\|, \quad \forall\, t \geq t_0. \tag{5}$$

⌟

IV. An Operator for stability analysis

To analyze nonlinear system stability from a differential viewpoint, we study the evolution of the variational state along the system trajectory. Under suitable invertibility conditions, the variational dynamics can be written in new coordinates via the matrix pair (T, Λ) or $(\mathbb{T}, \mathbb{A})$, providing two equivalent closed-form expressions for the state transition matrix Φ of the variational dynamics.

A. The operator spectral decomposition

Theorem IV-A.1. *Consider system* (2)*, with $f \in \mathcal{C}^1(\mathcal{X})$ on the forward invariant set $\mathcal{X}$ and its variational dynamics* (3) *in $\mathbb{R}^n$. Given the $\mathcal{C}^1$ vector-valued map f, define the operator $\mathscr{A} : \mathcal{C}^1(\mathcal{X}) \to \mathcal{C}^0(\mathcal{X})$ as*

$$\mathscr{A}[\cdot] := \frac{d[\cdot]}{dt} + [\cdot]\frac{\partial f(x)}{\partial x}. \tag{6}$$

Assume there exist n eigenfunctions $T_j \in \mathcal{C}^1(\mathcal{X})$ of $\mathscr{A}$, $j = 1, \ldots, n$, pointwise linearly independent, associated with eigenvalues $\lambda_j \in \mathcal{C}^0(\mathcal{X})$ with $\lambda_j : \mathcal{X} \to \mathbb{C}$, $j = 1, \ldots, n$, defined according to Def. II-A.2. Then, the transition matrix $\Phi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ of the variational dynamics (3) *reads as*

$$\Phi(t, t_0) = T(x)^{-1} \exp\left(\int_{t_0}^{t} \Lambda(x(\tau))d\tau\right) T(x_0) \tag{7}$$

where $\Lambda(x) = \mathrm{diag}\big(\lambda_1(x), \ldots, \lambda_n(x)\big) \in \mathbb{C}^{n \times n}$ and $T^\top = \big[T_1^\top \mid \cdots \mid T_n^\top\big] \in \mathbb{C}^{n \times n}$. ⌟

Proof. The proof relies on the diagonalization of the variational dynamics (3) via the change of coordinates $\delta z_j(t) := T_j(x(t))\delta x(t)$, $\delta z_j \in \mathbb{C}$, $j = 1, \ldots, n$. By definition, T_j is the left eigenfunction of the operator $\mathscr{A}$ associated with the eigenvalue λ_j and thus satisfies, $\forall x \in \mathcal{X}$, $j = 1, \ldots, n$,

$$\frac{d\,T_j(x)}{dt} + T_j(x)\,\frac{\partial f(x)}{\partial x} = \lambda_j(x)\,T_j(x).$$

Taking the time derivative of $\delta z_j(t)$ gives, for $j = 1, \ldots, n$

$$\dot{\delta z}_j(t) = \lambda_j(x)\,\delta z_j(t).$$

Since, in the new coordinates, the system is diagonal (since, each state evolution is independent from one another), linear, and time-varying, its closed form solution is

$$\delta z_j(t) = \exp\left(\int_{t_0}^{t} \lambda_j(x(\tau))d\tau\right) \delta z_j(t_0)$$

with $\delta z_j(t_0) = T_j(x_0)\,\delta x_0$, for $j = 1, \ldots, n$.
Finally, collecting the scalar equations into a vectorial form $\delta z(t) := \Big(\delta z_1(t), \ldots, \delta z_n(t)\Big) \in \mathbb{C}^n$, and employing the pointwise invertibility property of T, we conclude (7). ☐

Comparison with the standard eigenproblem

Let us denote a trajectory of system (2) as $x(t) = \varphi(x_0, t; t_0)$, and set, for notational simplicity,

$$A(x) := \frac{\partial f(x)}{\partial x} \in \mathbb{R}^{n\times n}.$$

Writing explicitly the operator eigenproblem $\mathscr{A}[T_j] = \lambda_j\ T_j$, yields the (left eigenfunction) explicit relationship

$$\frac{d\,T_j(x)}{dt} + T_j(x)\ A(x) = \lambda_j(x)\ T_j(x), \quad \forall x \in \mathcal{X}, \quad (8)$$

where the pair (λ_j, T_j) is referred to as the *dynamical eigenpair*. Furthermore, let $(\mu_j(x), v_j(x)) \in \mathbb{C}\times\mathbb{C}^{1\times n}$, with $\mu_j \in \mathcal{C}^0$ and $v_j \in \mathcal{C}^0$, denote a classical left eigenpair of $A(x)$, i.e., the solution of the standard eigenproblem

$$v_j(x)\ A(x) = \mu_j(x)\ v_j(x), \quad \forall x \in \mathcal{X}. \quad (9)$$

Comparing the two expressions (8) and (9), we see that the standard left eigenpair (μ_j, v_j) does not generally solve the operator eigenproblem $\mathscr{A}[T_j] = \lambda_j T_j$ unless $\dot{T}_j$ is identically zero, i.e., unless $v_j = T_j$ is a constant vector (as in the LTI systems case). Rewriting (8) as

$$\dot{T}_j(x) = T_j(x)\Big(\lambda_j(x) I - A(x)\Big) \quad (10)$$

we observe that, for a prescribed function λ_j, the map T_j satisfies another LTV ordinary differential equation (ODE).

B. A more general transformation

The eigenproblem associated with the operator $\mathscr{A}$ defined in (6), i.e., finding n eigenpairs (λ_j, T_j) such that the left eigenfunction T_j, $j = 1, \ldots, n$, are pointwise linearly independent, can be expressed in matrix form using the matrices $\Lambda, T \in \mathbb{C}^{n\times n}$ as in Th.IV-A.1, i.e.,

$$\dot{T}(x) + T(x)\ A(x) = \Lambda(x) T(x), \quad (11)$$

with $T(x) \in \mathcal{C}^1$ pointwise invertible and $\Lambda \in \mathcal{C}^0$.

It is worth noting that for the stability analysis of the variational dynamics, or for a closed-form expression of the variational state transition matrix $\Phi(t, t_0)$ such that $\delta x(t) = \Phi(t, t_0)\delta x_0$, it is not necessary for the new coordinates $\delta z := T(x)\,\delta x$ to be independent. In particular, we have the following generalization.

Theorem IV-B.1. *Consider system* (2)*, with* $f \in \mathcal{C}^1(\mathcal{X})$ *on the region* $\mathcal{X}$ *and its variational dynamics* (3)*. Define real-valued matrix functions* $\mathbb{T}, \mathbb{A} : \mathcal{X} \to \mathbb{R}^{n\times n}$*, with* $\mathbb{T} \in \mathcal{C}^1(\mathcal{X})$ *and* $\mathbb{A} \in \mathcal{C}^0(\mathcal{X})$*. Assume that, for any* $x_a, x_b \in \mathcal{X}$*,* $\mathbb{A}$ *satisfies the commutativity condition*

$$\mathbb{A}(x_a)\mathbb{A}(x_b) = \mathbb{A}(x_b)\mathbb{A}(x_a) \quad (12)$$

and that $(\mathbb{T}, \mathbb{A})$ *solves the matrix ODE*

$$\dot{\mathbb{T}}(x) + \mathbb{T}(x)\frac{\partial f}{\partial x}(x) = \mathbb{A}(x)\mathbb{T}(x), \quad \forall t \geq t_0 \quad (13)$$

with $\mathbb{T}(x)$ *pointwise invertible on* $\mathcal{X}$*. Then the explicit solution of the variational state* δx*, for all* $t \geq t_0$*, is*

$$\delta x(t) = \mathbb{T}(x)^{-1} \exp\left(\int_{t_0}^{t} \mathbb{A}(x(\tau))ds\right)\ \mathbb{T}(x_0)\ \delta x(t_0). \quad (14)$$

⌟

Proof. The solution pair $(\mathbb{T}, \mathbb{A})$, with $\mathbb{T}(x)$ everywhere invertible, allows us to define the change of coordinates $\delta z = \mathbb{T}(x)\delta x$, whose dynamics is

$$\dot{\delta z} = \mathbb{A}(x)\delta z, \qquad \delta z(t_0) = \mathbb{T}(x_0)\delta x_0. \quad (15)$$

Thanks to the commutativity assumption (12), the linear time-varying system (15) admits a closed form solution

$$\delta z(t) = \exp\Big(\int_{t_0}^{t} \mathbb{A}(x(\tau))\ d\tau\Big)\delta z(t_0)$$

from which

$$\delta x(t) = \mathbb{T}(x)^{-1}\ \exp\Big(\int_{t_0}^{t} \mathbb{A}(x(\tau))\ d\tau\Big)\mathbb{T}(x_0)\delta x_0,$$

which proves the claim. □

Equation (14) provides another closed form expression for the state transition matrix of the variational dynamics

$$\Phi(t, t_0) = \mathbb{T}(x)^{-1} \exp\Big(\int_{t_0}^{t} \mathbb{A}(x(\tau))d\tau\Big)\mathbb{T}(x_0), \quad (16)$$

which is equivalent to (7).

Remark IV.1 Equation (16) shows that the time evolution of $\delta x(t)$ can be *decomposed* into contributions from $\mathbb{T}(x)$ and from $\mathbb{A}(x)$, which are *dynamically* interconnected with the Jacobian of $f(x)$ through the matrix equation (13). ⌟

This decomposition generalizes the standard Jordan eigenproblem associated with A, for LTI systems, for which it is well known that constant eigenvectors T always provide an admissible solution to the eigenproblem associated with $\mathscr{A}$. Furthermore, the matrix pair $(\mathbb{T}, \mathbb{A})$ can be seen as a generalization of a real transformation matrix and a real Jordan form realization of a constant state matrix A with complex eigenvalues.

Solving (13) with constraint (12) is less restrictive than solving the full dynamical left eigenproblem (11) because:

- it removes the limitation for the variational dynamics to be diagonal in the new coordinates;
- for time invariant vector fields, if $\mathbb{A}(x)$ is diagonal[2], the i-th row of $\mathbb{T}(x)$, i.e., $\mathbb{T}_i(x)$, associated with the i-th diagonal term of $\mathbb{A}$, for $i = 1, \ldots, n$, it requires a weaker differentiability condition than the nonlinear left eigenfunctions in (1). Only differentiability in time is needed; see the discussion on the solution of the regulator equations in [23];
- it does not directly require the integrability property in Theorem II-B.1.

For completeness, the self-commutativity property (12) is satisfied by: constant matrices, Jordan blocks with time-varying diagonal matrices, matrices sharing the same eigenvector basis or simultaneously triangularizable [15], polynomials or analytic functions of the same matrix [12], circulant matrices [10], and, more generally, elements of the same commutative subalgebra [7].

[2]By allowing complex-valued solutions $(\mathbb{T}, \mathbb{A})$, i.e., $\mathbb{T}, \mathbb{A} : \mathcal{X} \to \mathbb{C}^{n\times n}$.

V. Differential exponential stability criteria

Under the assumption that system trajectories exist for all $t \in [t_0, \infty)$, which is ensured by the forward completeness of system dynamics, the following result characterizes the differential regional exponential stability of the nonlinear system (2).

Theorem V-.1 (Differential Regional Exponential Stability). *Under the assumption of Theorem IV-B.1, if one of the two following conditions holds, the system is differentially exponentially stable for all $x_0 \in \mathcal{X}_0 \subseteq \mathcal{X}$:*

a) for every $x_0 \in \mathcal{X}_0$, there exist $\underline{\sigma}, \underline{\alpha} \in \mathbb{R}_{>0}$, with $\underline{\sigma} = \underline{\sigma}_{x_0}$ and $\underline{\alpha} = \underline{\alpha}_{x_0}$, such that $\sigma_{\min}(\mathbb{T}(x)) \geq \underline{\sigma} > 0$ for all $x \in \mathcal{X}$, and

$$\int_{t_0}^{t} \beta(x(\tau))d\tau \geq (t - t_0)\ \underline{\alpha}\,, \quad \forall t \in [t_0, \infty);$$

b) for every $x_0 \in \mathcal{X}_0$, there exists a positive real $\underline{\alpha} = \underline{\alpha}_{x_0}$ such that, for all $x \in \mathcal{X}$,

$$\beta(x) + \frac{d}{dt} \ln \Big[\sigma_{\min}\big(\mathbb{T}(x)\big)\Big] \geq \underline{\alpha} > 0; \tag{17}$$

where the function $\beta : \mathcal{X} \to \mathbb{R}$ is defined as $\beta(x) = -\lambda_{\max}(\mathrm{sym}(\mathbb{A}(x)))$. ⌟

Proof. The system is forward complete, hence the trajectory $x(t) = \varphi(x_0, t; t_0)$ exists for every $t \geq t_0$. Thus, the variational dynamics along $x(t)$ and the associated state-transition matrix $\Phi(t, t_0)$ are well defined on $[t_0, \infty)$. Using the matrix pair $(\mathbb{T}, \mathbb{A})$, we express the state transition matrix of the system as in (16) and bound its norm as

$$\begin{aligned}\|\Phi(t, t_0)\| &\leq \|\mathbb{T}(x)^{-1}\| \Big\| \exp\Big(\int_{t_0}^{t} \mathbb{A}(x(\tau))d\tau\Big)\Big\| \|\mathbb{T}(x_0)\| \\ &\leq \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\sigma_{\min}\big(\mathbb{T}(x)\big)} \exp\Big(-\int_{t_0}^{t} \beta(x(\tau))\,d\tau\Big).\end{aligned}$$

From this upper bound, we independently verify each sufficient condition.
To this end, suppose that $a)$ holds. Then the upper bound of the state transition matrix reads as

$$\begin{aligned}\|\Phi(t, t_0)\| &\leq \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\sigma_{\min}\big(\mathbb{T}(x)\big)} \exp\Big(-\int_{t_0}^{t} \beta(x(\tau))d\tau\Big) \\ &\leq \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\underline{\sigma}} \exp\Big(-\underline{\alpha}(t - t_0)\Big).\end{aligned}$$

Then from the state evolution $\delta x(t) = \Phi(t, t_0)\delta x_0$ and the upper bound of its norm $\|\delta x(t)\| \leq \|\Phi(t, t_0)\|\|\delta x_0\|$, we see that the state trajectory satisfies the exponential stability (5) with

$$\kappa = \kappa_{x_0} = \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\underline{\sigma}_{x_0}}, \quad \text{and} \quad \alpha = \alpha_{x_0} = \underline{\alpha}_{x_0},$$

thus showing the first part of the result.

Suppose now that conditions $b)$ hold. Then, by taking the natural logarithm of the upper bound of the state transition matrix, we obtain

$$\begin{aligned}\ln\|\Phi(t, t_0)\| &\leq \ln \sigma_{\max}\big(\mathbb{T}(x_0)\big) - \ln \sigma_{\min}\big(\mathbb{T}(t)\big) \\ &\quad - \int_{t_0}^{t} \beta(x(\tau))d\tau \\ &\leq \ln \sigma_{\max}\big(\mathbb{T}(x_0)\big) - \ln \sigma_{\min}\big(\mathbb{T}(x_0)\big) \\ &\quad - \int_{t_0}^{t} \beta(x(\tau)) + \frac{d}{d\tau} \ln \big[\sigma_{\min}\big(\mathbb{T}(x(\tau))\big)\big]\, d\tau.\end{aligned}$$

where in the last step, we use the fundamental theorem of calculus to merge the contributions induced by $\mathbb{A}$ and those induced by the rate of change of the transformation $\mathbb{T}$. Exponentiating both sides, we derive the upper bound

$$\begin{aligned}\|\Phi(t, t_0)\| &\leq \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\sigma_{\min}\big(\mathbb{T}(x_0)\big)} \cdot \\ &\quad \cdot \exp\Big(-\int_{t_0}^{t} \beta(x(\tau)) + \frac{d}{d\tau} \ln \big[\sigma_{\min}\big(\mathbb{T}(\tau)\big]\, d\tau\Big).\end{aligned}$$

Now, under condition (17), the exponential term can be simplified by its upper bound, yielding

$$\|\Phi(t, t_0)\| \leq \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\sigma_{\min}\big(\mathbb{T}(x_0)\big)} \exp\Big(-\underline{\alpha}(t - t_0)\Big).$$

Therefore, from the upper bound of the evolution of the state $\|\delta x(t)\| \leq \|\Phi(t, t_0)\|\|\delta x_0\|$, we retrieve the definition of exponential stability (5) where

$$\kappa = \kappa_{x_0} = \frac{\sigma_{\max}\big(\mathbb{T}(x_0)\big)}{\sigma_{\min}\big(\mathbb{T}(x_0)\big)}, \quad \text{and} \quad \alpha = \alpha_{x_0} = \underline{\alpha}_{x_0}.$$

This concludes the proof. □

The interested reader is referred to [28] for a comparison between the proposed approach and the contraction-theory methodology.

VI. Example

Consider the nonlinear homogeneous system in [18, ex.8], with dynamics

$$\begin{aligned}\dot{x}_1 &= -x_2(-1 + 4x_1 + 4x_2 - 4x_2^2) \\ \dot{x}_2 &= -2(x_1 + x_2 - x_2^2)\end{aligned}\,, \tag{18}$$

and Jacobian matrix given by

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} -4x_2 & 1 - 8x_2 + 12x_2^2 - 4x_1 \\ -2 & -2 + 4x_2 \end{bmatrix}.$$

In our case, by applying the method proposed in Sec. IV-B, we obtain a real-valued solution pair $(\mathbb{T}, \mathbb{A})$ to (13) given by

$$\mathbb{T}(x) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} - 2x_2 \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

It is worth noticing that, in this case, $\mathbb{A}$ is not diagonal but constant and, therefore, satisfies the commutativity condition (12). Applying the closed-form solution in (16), we upper bound the evolution of the variational state as

$$\|\delta x(t)\|^2 \leq \frac{\sigma_{\max}(\mathbb{T}(x_0))}{\sigma_{\min}(\mathbb{T}(x))} \exp(-2t)\|\delta x_0\|^2,$$

where

$$\sigma_{\min}(\mathbb{T}) = \frac{1 + 2\big(2x_2 - \frac{1}{2}\big)^2 - \sqrt{\Big(1 + 2\big(2x_2 - \frac{1}{2}\big)^2\Big)^2 - 1}}{2},$$
$$\sigma_{\max}(\mathbb{T}) = \frac{1 + 2\big(2x_2 - \frac{1}{2}\big)^2 + \sqrt{\Big(1 + 2\big(2x_2 - \frac{1}{2}\big)^2\Big)^2 - 1}}{2}.$$

A straightforward analysis shows that $\sigma_{\min}(\mathbb{T}(x)) \xrightarrow[|x_2|\to\infty]{} 0^+$, implying $\inf_{x\in\mathbb{R}^2} \sigma_{\min}(\mathbb{T}(x)) = 0$. Moreover, its maximum is achieved at the points $(x_1^\star, x_2^\star)$ with $x_2^\star = \pm 1/2$, yielding $\max_{x\in\mathbb{R}^2} \sigma_{\min}(\mathbb{T}(x)) = 1/2$. However, $\sigma_{\min}(\mathbb{T}(x))$ does not admit a uniform positive lower bound, but can be lower bounded based on the system's initial condition or on the forward invariant region of interest $\mathcal{X} \subset \mathbb{R}^2$. To this end, consider a positive real valued lower bound map $\underline{\lambda}(x_0)$ defined for any $x_0 \in \mathcal{X}$, such that $\underline{\lambda}(x_0) \leq \sigma_{\min}(\mathbb{T}(x(t))$, for all $t \geq t_0$, then

$$\|\delta x(t)\| \leq \sqrt{\frac{\sigma_{\max}(\mathbb{T}(x_0))}{\underline{\lambda}(x_0)}} \exp(-t)\|\delta x(0)\|.$$

Since $\sigma_{\max}(\mathbb{T}(x_0))$ is upper bounded by a real scalar at any $x_0 \in \mathbb{R}^2$, we conclude that the system (18) is differentially semi-globally exponentially stable, with a non-uniform scaling factor κ. Equivalently, this established differential regional exponential stability with fixed bounds. Note that Kawano et al. in [18] proved the global asymptotic stability of the system (18). The interested reader is referred to [28] to compare this example with a contraction-based methodology.

VII. Conclusions

In this work, we examine the differential stability properties through the lens of a spectral decomposition of a newly introduced operator. The collection of derived eigenfunctions provides a change of coordinates, from which we derive a criterion for differential exponential stability. For completeness of exposition, we also provide a comparison between the novel operator spectral decomposition and the classical Jacobian-based matrix decomposition. Furthermore, we employed a weaker version of the eigenproblem formulation to provide two differential exponential stability criteria. To conclude, we consider a nonlinear example to illustrate the effectiveness of the proposed framework.

This work lays the groundwork for a unified differential approach to stability, providing new tools to study nonlinear, time-varying, and non-diagonalizable dynamical systems and highlighting several promising directions for future research. Extension of this work [26]–[28] are currently under review, where a more general framework is investigated, alternative stability criteria, a connection with contraction theory, and future directions can be found.

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