On the Role of Natural Frequency ω_n in the Overshoot of σ - ω_n Second Order Transfer Functions with Constant Damping Factor: Application to a Robotic Position Servomotor

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***Abstract*— Classic control design begins with a mathematical model of the physical plant to be controlled which is assumed linear time invariant, so modeled through a transfer function.**

This paper considers rational transfer functions with real coefficients in the complex variable s. Specifically it deals with strictly proper second-order ones with the following ad-hoc σ - ω_n structure:

$$\frac{\omega_n^2}{s^2 + 2\sigma s + \omega_n^2}, \quad \sigma, \omega_n > 0. \tag{1}$$

It is proven that despite a fixed and constant $\sigma > 0$ called *'damping factor'* or *'attenuation'*, it might be possible to alter or vary (with restrictions) simultaneously both the overshoot and time speed response by modifying only the *natural frequency* $\omega_n > 0$. In addition, an explicit upper bound upon ω_n is derived to ensure an overdamped response, thereby avoiding both oscillations and overshoot. Theoretical results were examined through numerical simulations and experimentally validated using a Sallen-Key low-pass filter circuit that emulates a position-controlled servomotor model.

I. Introduction

Standard automatic controller design initiates from a mathematical model of the physical plant to be controlled, this first key step is called modeling [1]. In general form, these models are described by differential equations established in terms of input and output variables. When dealing with Linear Time Invariant (LTI) plants, the typical modeling tool is the transfer function inherited from the Laplace transform $\mathcal{L}\{\cdot\}$ which is the foundation of classic control.

Second-order transfer functions are commonly studied and used in automatic control systems [2], [3], for instance open-loop models like simplified DC motors [4], RLC electric circuits [3], [5], Spring-Mass-Damper mechanical systems [2, p. 264], and simplified automotive suspension systems [5]. Also closed-loop systems such as Proportional control of armature controlled DC motors —position servo actuators— [4] are well modeled by second-order transfer functions.

It is shown that the overshoot, as defined here for the "σ - ω_n second-order transfer functions"

$$\frac{\omega_n^2}{s^2 + 2\sigma\, s + \omega_n^2}, \tag{2}$$

can be altered without explicitly adjusting neither the so-called damping ratio ζ nor the damping factor σ, but adjusting only the natural frequency ω_n.

Transfer function (2) matches the model of a class of position servo actuators [4], where the natural frequency ω_n is a linear function of the inner proportional gain.

This paper offers an alternative perspective to that found in classical textbooks on second-order linear system response analysis, focusing on variations in the natural frequency parameter rather than the damping ratio to govern the overshoot. The theory is supported by experimental results obtained using a Sallen-Key low-pass filter circuit that emulates the behavior of a position servo actuator.

II. Preliminaries

This study considers rational transfer functions with real coefficients in the complex variable s. Throughout the paper, the symbols u and y denote the time-dependent input and output of the system, respectively. Two particular structures of second-order transfer functions are recalled below.

A. ζ - ω_n second-order form

The strictly proper and relative degree equal to two, second-order transfer function with the well accepted *normal* or *standard* ζ - ω_n form structure and notation is given by [2, p. 226]:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}; \quad \zeta \geq 0; \quad \omega_n > 0. \tag{3}$$

The constant $\zeta \geq 0$ is called *'damping ratio'*, and $\omega_n > 0$ is the *'natural frequency'*, [3, p. 169].

Remark 1. According to values of the damping ratio $\zeta \geq 0$, several kinds of systems result, such as: uderdamped, overdamped and critically damped. The larger ω_n, the faster the response; the smaller ζ, the larger the overshoot [5, p. 120]. It is well known that if ζ is increased beyond the unity, the response becomes overdamped and it will not oscillate [2, p. 227]. However, $1 > \zeta \geq 0$ generates overshoot and oscillations when a step input is supplied. In sum, only numerical values of ζ governs the overshoot of system (3) irrespective of a fixed ω_n.

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B. σ-ω_n second-order form

Above transfer function (3) can be rewritten in the alternative *constant damping factor* σ or σ - ω_n form of interest in this paper:

$$\frac{\omega_n^2}{s^2 + 2\sigma\, s + \omega_n^2}, \tag{4}$$

where $\sigma \triangleq \zeta\omega_n$ is called the *'damping factor'* [5, p.116], or named *'damping constant'* in [6, p.327], or *'attenuation'* in [2, p.225].

Remark 2. It is important to emphasize that, aside from having different names and symbols, the damping ratio ζ, and the damping factor σ, they are also different conceptual items.

The name and domain of the parameters in the transfer functions (3) and (4), as considered in this paper, are listed in Table I.

TABLE I

PARAMETERS OF SECOND-ORDER TRANSFER FUNCTIONS (3) AND (4).

Name	Symbol	Domain	Unit
Natural frequency	ω_n	> 0	rad/sec
Damping **ratio**	ζ	> 0	dimensionless
Damping **factor** = attenuation	$\sigma{=}\zeta\,\omega_n$	> 0	rad/sec

III. MAIN RESULTS

Unlike the well known ζ - ω_n system (3) where the overshoot depends only upon the numerical value of its parameter 'damping ratio' ζ, this paper claims that for σ - ω_n system (4), the overshoot is not determined by particular values of neither σ nor ω_n, but by the relationship between them: in order to avoid overshoot such a relation is $\omega_n/\sigma \leq 1$.

The main result regarding the absence of overshoot as a function of the natural frequency ω_n in the σ - ω_n transfer function (4) to a step exogenous input is established in the following proposition. This proposition provides admissible values for the natural frequency $\omega_n > 0$ relative to σ that ensures a system response y free from overshoot.

Proposition 1. Consider the σ - ω_n second-order transfer function (4):

$$\frac{\omega_n^2}{s^2 + 2\sigma\, s + \omega_n^2}. \tag{5}$$

Let u and y be its input and output variables, respectively.

It is assumed that the **damping factor** $\sigma > 0$ is fixed, constant, and exactly known. The latter can be relaxed by the less stringent assumption of a known lower bound $\bar{\sigma} > 0$ upon the true σ, i.e., $\sigma > \bar{\sigma} > 0$. Constant σ can not be modified. The **natural frequency** $\omega_n > 0$ is the only modifiable —adjustable— parameter available for the designer.

Regardless of the numerical strictly positive values of the two parameters σ and ω_n, the steady state of the output y in system (5) matches any constant input $u = u_0 \in \mathbb{R}$ [4], i.e.:

$$\lim_{t\to\infty} y(t) = u_0.$$

However, the time response $y(t)$ may present undesirable overshoot during the transient before tending asymptotically to the constant value u_0. The values of the natural frequency $\omega_n > 0$ for achieving a system response y free from overshoot are:

$$\omega_n \in (0, \sigma).$$

***Proof*.** Since ω_n is within the interval $(0, \sigma) \subset \mathbb{R}$, then it must satisfy:

$$\omega_n < \sigma.$$

But recalling definition of the damping factor $\sigma = \zeta\omega_n$, it yields

$$\omega_n < \underbrace{\zeta\omega_n}_{\sigma},$$

which in its turn implies:

$$1 < \zeta.$$

It finally means that the system is overdamped [2, p. 227], thus no overshoot will be present in the system (5) response y to step inputs u.

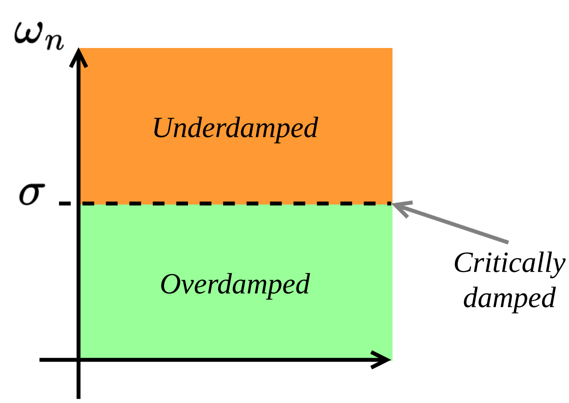

Fig. 1. Relationship between damping factor σ - natural frequency ω_n and resulting type of σ - ω_n second-order system (5) .

Remark 3. Figure 1 summarizes in a graphic way the kind of σ - ω_n second-order system (5) according to the relationship between the natural frequency $\omega_n > 0$ and the constant and unchanging damping factor $\sigma > 0$. In words:

- $\omega_n < \sigma$: overdamped system,
- $\omega_n > \sigma$: underdamped system,
- $\omega_n = \sigma$: critically damped.

The features of above classes of second-order systems are well described in basic textbooks on automatic control, for instance see [2], [3], [6], [7]. At this moment one mentions that a remarkable characteristic of second-order systems, both, critically damped ($\zeta = 1$ or $\omega_n = \sigma$), and overdamped ($\zeta > 1$ or $\omega_n < \sigma$) is that they do not present any oscillation for step input excitations.

IV. Numerical Simulation

As a numerical example consider the transfer function:

$$\frac{{\omega_n}^2}{s^2 + 20\,s + {\omega_n}^2}. \tag{6}$$

Thus the damping factor is $\sigma = 10$ because $2\sigma = 20$. According to Proposition 1, the response y to step inputs will be free from overshoot provided that $0 < \omega_n < \sigma$, namely if

$$10 > \omega_n > 0.$$

Numerical simulations using unit step input $u = u_0 = 1$ for the following set of 3 values $\omega_n \in \{5, 9, 25\}$ were performed. As expected, in plots of Fig . 2, no overshoot appears for the two smaller values than $\omega_n < \sigma = 10$, i.e., for $\omega_n = 5, 9$. However, overshoot is present for large enough ω_n, in the sense $\omega_n = 25 > \sigma = 10$. Thus, it shows that by manipulating only the natural frequency ω_n, both the response speed and the overshoot are altered.

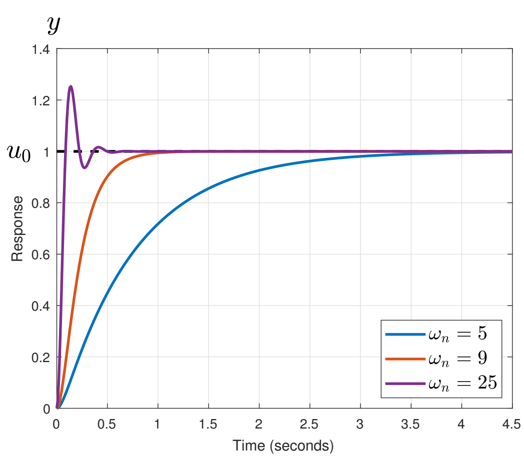

Fig. 2. Numerical simulation results to step input in system (6).

V. Analogical simulation: a position servo actuator

A. Transfer function of a Sallen-Key low-pass filter

To validate the theoretical results presented in Section III, an experimental test is conducted using a Sallen-Key low-pass filter, which can be modeled by a second-order transfer function as described below [8]:

$$\frac{E_o(s)}{E_i(s)} = A\frac{\frac{1}{R_1R_2C_1C_2}}{s^2 + \left(\frac{1}{R_2C_1} + \frac{1}{R_1C_1} + \frac{1-A}{R_2C_2}\right)s + \frac{1}{R_1R_2C_1C_2}}, \tag{7}$$

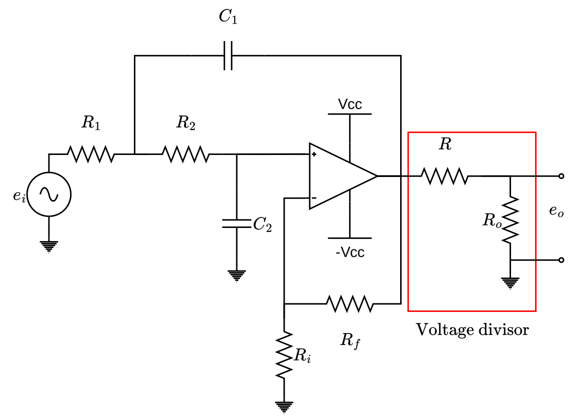

Fig. 3. Sallen and Key low-pass filter schematic diagram.

where $E_i(s)$ and $E_o(s)$ denote the Laplace transform of the input and output voltages of the circuit, respectively; R_1, R_2 are resistors, C_1, C_2 are capacitors, and A represents the gain of a non-inverting amplifier given by

$$A = 1 + \frac{R_f}{R_i}, \tag{8}$$

here, R_f and R_i refer to resistors in the circuit. Figure 3 shows the schematic diagram of the Sallen-Key low-pass filter used in this study. In the figure, R and R_o are resistors that form a voltage divider at the output of the operational amplifier. The operational amplifier (OpAmp) used in the experimental tests was a TL082 commercial integrated circuit, with a dual supply voltage $V_{cc} = \pm 14$V.

Considering the parameter values $R_1 = 575.8$ kΩ, $R_2 = 576.8$ kΩ, $C_1 = 10.49$ nF, $C_2 = 10.18$ nF, $R_f = 10.5$ kΩ, $R_i = 9.86$ kΩ, $R = 9.85$ kΩ, $R_o = 9.15$ kΩ, the transfer function given in equation (7) can be expressed in its numerical form as:

$$G(s) = \frac{2.8196 \times 10^4}{s^2 + 149.4719\,s + 2.8196 \times 10^4}, \tag{9}$$

with $\omega_n = 167.9152$, $\zeta = 0.4451$, and $\sigma = 74.7360$. Values for components in the Sallen-Key low-pass filter were selected to reflect the dynamic behavior of a position servo-motor, allowing the second-order transfer function to imitate the system's response under typical operating conditions.

B. A 2nd order transfer function of a position servo actuator

The transfer function of a position servomotor can be represented in a structure consistent with equation (7) as follows [4]:

$$G(s) = \frac{\frac{k\,k_P}{J\,R}}{s^2 + \frac{(kk_b + Rb_0)}{JR}s + \frac{k\,k_P}{J\,R}}, \tag{10}$$

with R representing the internal permanent magnet DC (PMDC) motor armature resistance, J the rotor inertia, k_P denoting the inner proportional control gain, k_b the back electromotive force constant, k the torque constant, and b_0 the viscous friction coefficient. From reference [9], which presents the experimental validation of a model for the Dynamixel MT-28AT servomotor, the subsequent set of parameters was identified: $J = 8.68 \times 10^{-8}$ kgm^2, $R = 8.3\ \Omega$, $k = 0.0107$ Nm/A, $k_b = 0.0107$ Vs/rad, $b_0 = 8.87 \times 10^{-8}$ Nms/rad, and $k_P = 2$. Accordingly, the position servo actuator's transfer function, given in Equation (10), is numerically expressed as follows:

$$Gp(s) \triangleq \frac{2.9704 \times 10^4}{s^2 + 159.9387\, s + 2.9704 \times 10^4}, \qquad (11)$$

with $\omega_n = 172.3487$ (corresponding to a proportional gain k_P setting of 2), $\zeta = 0.4640$, and $\sigma = 79.9693$. Since $\omega_n > \sigma$, and in accordance with Remark 1, the system is underdamped and it exhibits an oscillatory response.

By comparing the quadratic and linear terms of the characteristic polynomials corresponding to the experimental transfer function (9) and the theoretical transfer function (11), it is observed that the deviation between the numerical coefficients yields an average discrepancy of approximately 1.3% for the independent term (ω_n^2), and 3.33% for the linear term (2σ). This numerical error can be further minimized by optimizing the selection of the component values in the Sallen-Key filter circuit.

C. Analogical simulations results

The Sallen-Key low-pass filter circuit corresponding to the second-order transfer function in Equation (9), which was used for the experiments, is shown in Figure 4.

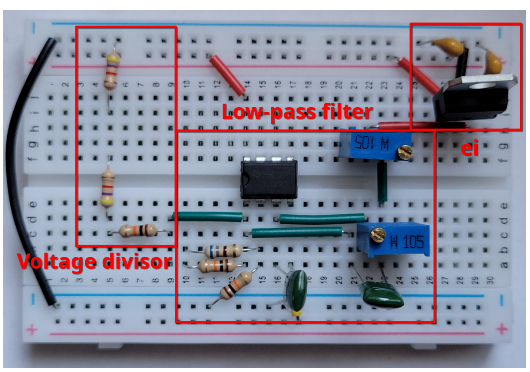

Fig. 4. Sallen-Key low-pass second-order filter electric circuit used in the experiment.

Figure 5 displays the graphical outcome of the experiment conducted on the Sallen-Key filter, with 5V reference voltage applied at e_i. The black dashed horizontal line represents the reference that the circuit output voltage (e_o) is expected to reach in steady state. The solid orange line illustrates the measured time evolution of the output voltage. The blue dashed line corresponds to the numerical simulation of the circuit performed using MATLAB R2024b, under the same conditions as the experiment. The results indicate that the output voltage reaches steady state at approximately 0.07s, aligning with the reference value.

During the transient period (before 0.07s), the system response is oscillatory. This behavior is consistent with numerical simulation results and demonstrates strong agreement between experimental observations and theoretical predictions. The presence of oscillations confirms that the system is underdamped, as the natural frequency value $\omega_n = 167.9152$ is significantly higher than the damping factor $\sigma = 74.7360$, satisfying the condition $\omega_n > \sigma$.

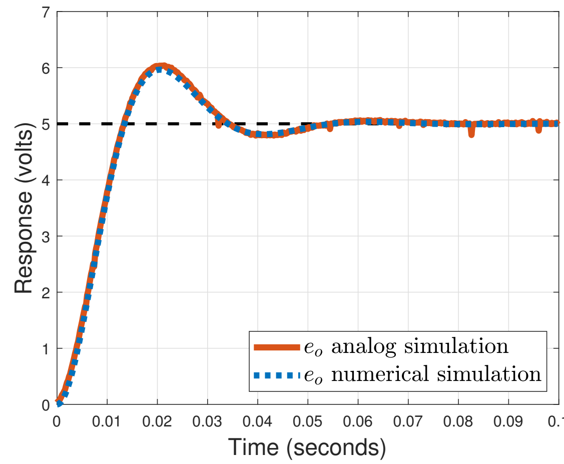

Fig. 5. Experimental result of the RLC electric circuit (9) simulating analogically the position servo actuator transfer function (11).

VI. Conclusions

With reference to σ - ω_n second-order transfer functions with relative degree equal to 2, it has been shown that for a fixed constant so-called 'attenuation parameter' σ defined by the damping ratio times the natural frequency, the overshoot of the system response to step inputs can be modified by adjusting only the natural frequency ω_n.

It has been proven that a sufficient condition for avoiding overshoot is $\sigma \geq \omega_n > 0$. This result could be applied in a class of position servo actuators —systems incorporating an inner adjustable proportional gain k_P controller for the internal armature-controlled PMDC motor— for tuning the embedded proportional gain k_P conductive to evade harmful overshoot. Qualitatively, the proportional gain should be sufficiently small.

A summary of the resulting type of system according to relationship between the attenuation parameter σ and the natural frequency ω_n is in Table II.

TABLE II
RESULTING SECOND-ORDER TYPE SYSTEM.

System type	Condition	Overshoot
Overdamped	$\omega_n < \sigma$	No
Underdamped	$\omega_n > \sigma$	Yes
Critically damped	$\omega_n = \sigma$	No

References

[1] K.J. Åström and R.M. Murray, *Feedback Systems: An introduction for Scientists and Engineers*. Princeton, USA: Princeton University Press, 2008.

[2] K. Ogata, *Modern Control Engineering,* Fourth Edition. New Jersey, USA: Prentice -Hall, Pearson Education International, 2002.

[3] N.S. Nise, *Control Systems Engineering*. Hoboken, NJ, USA: J. Wiley & Sons, 2015.

[4] R. Kelly, G. Zepeda, and C. Monroy, "On simplified models of position servo actuators for control purposes," in *IEEE Xplore, 2021 International Conference on Control, Automation and Diagnosis (ICCAD),* Grenoble, France, 3–5 Nov. 2021, pp. 1–6.

[5] Chi-T. Chen, *Analog & Digital Control System Design: Transfer-Function, State-Space, and Algebraic Methods*. Philadelphia, USA: Saunders College Publishing, 1993.

[6] R. C. Dorf and R.H. Bishop, *Modern Control Systems,* Fourteenth Edition, Global Edition. Harlow, CM17 9SR, United Kingdom: Pearson Education Limited, 2022.

[7] J.J. D'Azzo, C.H. Houpis, and S.N. Sheldon, *Linear Control Systems Analysis and Design with MATLAB,* Fifth Edition. New York, USA: Marcel Dekker Inc., 2003.

[8] F. James, *Operational amplifiers & linear integrated circuits: theory and application*. New York, USA: dissidents, 2018.

[9] M. Maximo, C. H. Ribeiro, and R. J. Afonso, "Modeling of a position servo used in robotics applications," in *Proceedings of the 2017 Simposio Brasileiro de Automaçao Inteligente (SBAI)*, 2017.