Monotonic Output Regulation of Discrete-Time MIMO Systems with Repeated Unit Zeros and Input Delays

Abishek Gourave and Tushar Jain

Abstract— **This paper studies monotonic output regulation for discrete-time LTI MIMO systems with multiple zeros at z = 1, with a primary focus on identifying its feasibility conditions. The analysis is then extended to systems with a d-step input delay, demonstrating that monotonicity can be achieved after d time steps. These two problems are linked through the common requirement of repeated pole placement, providing insight into the structural limitations of achieving a monotonic transient response. The theoretical results are verified on a discretized spring–mass system featuring multiple zeros at z = 1, demonstrating that with a d-step input delay, monotonic output regulation becomes achievable from the d-th step onward.**

I. Introduction

Safety and precision are foundational requirements in modern industrial process control. In many high-stakes applications, such as chemical engineering and precision manufacturing, the control objective extends beyond mere stability to encompass rigorous transient performance constraints. In these contexts, even a minor overshoot can violate strict tolerances, compromise product integrity, or trigger safety interlocks. Therefore, a desirable transient response must move beyond minimizing excursions and instead eliminate overshoot entirely through monotonic tracking.

[1] presents Moore's pole placement method for state feedback gain design K. [2] and [3] develop methods to design the gain K that ensures monotonic output convergence of the closed-loop system from any initial condition. This is done by hiding the eigenmodes from the output. [3] The assignment of stable zeros from the open-loop LTI system as closed-loop eigenvalues effectively hides eigenmodes from the output. K is designed such that any one eigenmode of the closed loop system remains visible in the outputs while the remaining eigenmodes are hidden, ensuring monotonicity. Furthermore, [2] shows that at least $n-p$ stable zeros are required to guarantee the monotonic convergence of each output to zero, where n, p are the number of states and output respectively. [3] also characterize the set of initial conditions that yield a non-overshooting response when output is superposition of multiple modes. In [4], a minimal-norm state feedback design is proposed to achieve monotonic output response. [5] formulate an LMI and an equality constraint, the solutions of which are used to construct the gain matrix K for a monotonic output response. In [6], the output regulation problem for MIMO systems is studied, where the reference and disturbance signal are generated by a linear autonomous system and output tracks the reference signal while rejecting disturbances. Observers are also designed for both state-feedback and output-feedback cases. In all the work monotonicity for the system having zeros on the unit circle is not discussed. In this work, we characterize the conditions for the existence of a state feedback gain K for discrete-time LTI systems with multiple zeros at one, under which the tracking error converges monotonically to zero while ensuring bounded control input. The proposed framework is then employed to track sinusoidal trajectories generated by a linear autonomous system. For simulation, a discretized spring–mass system is considered, which exhibits three zeros at one. Furthermore, a controller is designed after putting uniform input delay d in the same system.

The remainder of this paper is organized as follows. Section II, defines the problem. In Section III, the procedure for placing repeated zeros of the LTI system as closed loop eigenvalues is presented. Section IV gives the construction of controller for monotonic output regulation of system with a d-step input delay. In Section V, numerical example of discretized spring–mass system with multiple zeros at one are taken, and design a controller for tracking the sinusoidal signal generated by linear autonomous system and a controller is also design after putting uniform input delay.

II. Problem Formulation

In this paper we consider a discrete time linear multivariable systems ruled by the following equations:

$$\sum : \begin{cases} x(k+1) = Ax(k) + Bu(k) + Hw(k) \\ y(k) = Cx(k) + Du(k) \\ e(k) = Cx(k) + Du(k) + Fw(k) \end{cases} \tag{1}$$

Where, for all $k \geq 0$, the signal $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the measured output, and $e(k) \in \mathbb{R}^p$ is the regulated output. The exogenous signal $w(k) \in \mathbb{R}^q$ consists of a reference signal to be tracked and disturbance to be rejected.

$$w(k+1) = Sw(k), \qquad w(0) = w_0 \tag{2}$$

We also consider the nominal plant Σ_{nom}, which arises when the exosystem is excluded from consideration.

$$\sum_{nom} : \begin{cases} \tilde{x}(k+1) & = A\tilde{x}(k) + B\tilde{u}(k) \\ \tilde{e}(k) & = C\tilde{x}(k) + D\tilde{u}(k) \end{cases} \tag{3}$$

Abishek Gourave is with School of Computing and Electrical Engineering, Indian Institute of Technology Mandi, HP, India s21026@students.iitmandi.ac.in

T.Jain is with the Faculty of School of Computing and Electrical Engineering, Indian Institute of Technology Mandi, HP, India tushar@iitmandi.ac.in

Assumption 1. The following conditions are assumed to hold throughout this paper:

1.1 The eigenspectrum of S, denoted by $\sigma(S)$, satisfies $\sigma(S) \subseteq \{z \in \mathbb{C} : |z| \leq 1\}$, and all eigenvalues on the unit circle have equal algebraic and geometric multiplicities.
1.2 The matrices B and C are of full rank, respectively, which implies the absence of redundant control input and output measurements.
1.3 The pair (A, B) is stabilizable.
1.4 For all $\lambda \in \sigma(S)$, where $\sigma(S)$ denotes the spectrum of S, the following rank condition holds:

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = n+p.$$

1.5 The pair $\left(\begin{bmatrix} C & F \end{bmatrix}, \begin{bmatrix} A & H \\ 0 & S \end{bmatrix}\right)$ is detectable.
1.6 There exists matrices Γ and Π satisfying

$$\Pi S = A\Pi + B\Gamma + H$$
$$0 = C\Pi + D\Gamma + F$$

A. Transmission Zeros and Multiplicities

The algebraic and geometric multiplicities of transmission zeros can be characterized via both transfer function and state-space representations.

Definition 1 (Smith–McMillan Form [7]). Let $G(z) \in \mathbb{R}(z)^{p\times m}$ with normal rank r have the Smith–McMillan decomposition $G(z) = U(z)M(z)V(z)$, where $U(z), V(z)$ are unimodular, and $M(z) = \text{diag}\left(\frac{\varepsilon_1(z)}{\psi_1(z)}, \ldots, \frac{\varepsilon_r(z)}{\psi_r(z)}, 0, \ldots, 0\right)$ with coprime monic polynomials satisfying $\varepsilon_i(z) \mid \varepsilon_{i+1}(z)$ and $\psi_{i+1}(z) \mid \psi_i(z)$. A complex number $z_0 \in \mathbb{C}$ is a zero of $G(z)$ if $\text{rank}(G(z_0)) < r$ (i.e., $\varepsilon_i(z_0) = 0$ for some i). Its geometric and algebraic multiplicities are defined as:

$$m_g(z_0) = \text{card}\{i \mid \varepsilon_i(z_0) = 0\}, \qquad m_a(z_0) = \sum_{i=1}^{r} m_i(z_0)$$

where $m_i(z_0)$ is the multiplicity of z_0 as a root of $\varepsilon_i(z)$.

Definition 2 (State-Space Representation [8]). A scalar $\lambda \in \mathbb{C}$ is a transmission zero of the system if there exist non-trivial state-input vector pairs $\begin{bmatrix} v_{i,j}^T & u_{i,j}^T \end{bmatrix}^T$ that satisfy the Rosenbrock matrix equation for $j = 1$:

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_{i,1} \\ u_{i,1} \end{bmatrix} = \mathbf{0}, \quad i = 1, \ldots, m_g(\lambda)$$

where $m_g(\lambda)$ is the geometric multiplicity of λ. Here, the index i denotes the i-th linearly independent zero-vector, and $j = 1$ specifies the initial vector of the associated Jordan chain of generalized zero-vectors. $\begin{bmatrix} v_{i,1}^\top & u_{i,1}^\top \end{bmatrix}^\top$ are the primary zero directions. For $m_a(\lambda) > m_g(\lambda)$, the generalized zero directions form a Jordan chain solved recursively for $j = 2, \ldots, k_i$ via:

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_{i,j} \\ u_{i,j} \end{bmatrix} = \begin{bmatrix} v_{i,j-1} \\ \mathbf{0} \end{bmatrix}$$

The algebraic multiplicity is the total chain length: $m_a(\lambda) = \sum_{i=1}^{m_g(\lambda)} k_i$. This illustrates in Fig 1.

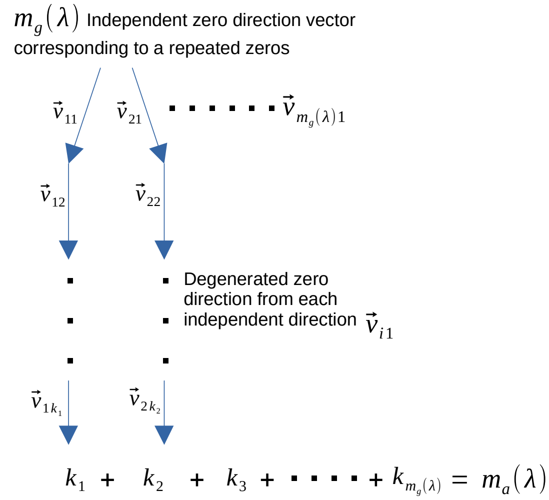

Fig. 1. Representing the zero direction for repeated zeros

Problem 1. If the nominal system (3) is characterized by repeated zeros at one, characterize feasibility conditions for which there exists a control input such that the tracking error of system (1) converges monotonically toward zero while the system states remain bounded.

Problem 2. Consider the system (1) with uniform input delay of d units.

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k-d) + Hw(k), \\ y_d(k) &= Cx(k) + Du(k-d) \\ e_d(k) &= Cx(k) + Du(k-d) + Fw(k), \end{cases} \tag{4}$$

Design a controller such that the tracking error $e_d(k)$ remains unaffected for the first d steps from the input and begins converging monotonically to zero for all $k \geq d$.

III. Placing Repeated Real Zero as Eigenvalue of the Closed Loop System

1) Real Repeated Zeros (λ_0): To embed a real transmission zero λ_0, with geometric multiplicity $m_g(\lambda_0)$ and algebraic multiplicity $m_a(\lambda_0)$, into a target closed-loop eigenvalue subset $\Lambda_{real} = \{\lambda_1, \ldots, \lambda_{m_a(\lambda_0)}\}$ where $\lambda_k = \lambda_0$. the primary zero directions ($j = 1$) are computed via:

$$\begin{bmatrix} A-\lambda_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_{i,1} \\ u_{i,1} \end{bmatrix} = \mathbf{0}, \quad i = 1, \ldots, m_g(\lambda_0) \tag{5}$$

For $m_a(\lambda_0) > 1$, the generalized directions in the Jordan chain are solved recursively for $j = 2, \ldots, k_i$:

$$\begin{bmatrix} A-\lambda_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_{i,j} \\ u_{i,j} \end{bmatrix} = \begin{bmatrix} v_{i,j-1} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

where $\sum_{i=1}^{m_g(\lambda_0)} k_i = m_a(\lambda_0)$. This yields the matrices $V_{re} = [v_{i,j}]$ and $U_{re} = [u_{i,j}]$.

2) Additional Pole Assignment and Gain Construction: The remaining $n - m_a(\lambda_0)$ eigenvalues $\Lambda_{rem} = \{\lambda_l\}$ are assigned by solving:

$$\begin{bmatrix} A-\lambda_l I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_l \\ u_l \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \beta_l \end{bmatrix} \tag{7}$$

where β_l is an arbitrary parameter vector (canonical vector to facilitate a monotonic response [6]), forming $V_{rem} = [v_l]$ and $U_{rem} = [u_l]$.

Defining $V = \begin{bmatrix} V_{re} & V_{rem} \end{bmatrix}$ and $U = \begin{bmatrix} U_{re} & U_{rem} \end{bmatrix}$, the feedback gain is $K = UV^{-1}$ (assuming V is non-singular), ensuring $\sigma(A+BK) = \Lambda_{real} \cup \Lambda_{rem}$.

When repeated system zeros are assigned as closed-loop eigenvalues, the resulting closed-loop matrix may become non-diagonalizable. Specifically, if the algebraic multiplicity $m_a(\lambda_0)$ exceeds the geometric multiplicity $m_g(\lambda_0)$, the closed-loop matrix is defective and contains non-trivial Jordan blocks [9](Ch.3). In discrete time systems, the presence of a Jordan block of size greater than one associated with zeros at unity is particularly critical. It induces polynomial growth in the state trajectory (proportional to $k, k^2, \ldots.$ where k is the discrete time step). This leads to unbounded state propagation and a subsequent failure to achieve internal stability.

Remarks 1. When one is a repeated system zero, assigning it as a closed loop eigenvalue in state-feedback design results in a bounded state trajectory if and only if its algebraic multiplicity equals its geometric multiplicity in closed loop, thereby avoiding the introduction of nontrivial Jordan blocks. The maximum number of zeros that can placed as closed-loop eigen value is r in case of real $z = \lambda_0$ as it is the maximum no. of zeros that can accommodate without repeating itself, where r is the normal rank of the transfer function matrix $G(z)$

IV. Constructing state feedback matrix for LTI system having fixed delay

Consider an LTI system given in (4) with uniform input delay d. From [10], this system can be represented in an augmented state-space form as

$$\begin{aligned} X(k+1) &= \bar{A}X(k) + \bar{B}u(k) + \bar{H}w(k), \\ e(k) &= \bar{C}X(k) + \bar{D}u(k) + Fw(k), \end{aligned} \tag{8}$$

where the augmented state is defined as

$$X(k) = \begin{bmatrix} x(k) \\ u(k-d) \\ u(k-d+1) \\ \vdots \\ u(k-1) \end{bmatrix} \tag{9}$$

and the system matrices are given by

$$\bar{A} = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & I_m \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & D & \cdots & 0 \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\bar{D} = 0$

The work in [2][5] establishes the necessary and sufficient conditions for achieving monotonic output tracking in discrete-time MIMO LTI systems. Specifically, it is shown that a state-feedback matrix can be constructed to ensure monotonic convergence in each output channel if and only if the system possesses $n-p$ stable transmission zeros.

Theorem 1. *If the delay-free system* (1) *admits a monotonic output regulation, then the corresponding delayed system* (4) *with uniform input delay d admits a monotonic output regulation after d steps.*

Proof: Let the nominal transfer function be $G(z) = L(z)M(z)R(z)$ in Smith-McMillan form, where $L(z)$ and $R(z)$ are unimodular matrices. Introducing uniform input delay $D(z) = z^{-d}I_m$ yields the delayed system transfer function $G_d(z) = L(z)M(z)R(z)D(z)$. Since $D(z)$ is a diagonal matrix with identical diagonal entries, it commutes with $R(z)$, yielding, $R(z)D(z) = D(z)R(z)$. This simplifies to:

$$G_d(z) = L(z)M(z)D(z)R(z) \tag{10}$$

Post-multiplying $M(z)$ by $D(z)$ increases the relative degree of its diagonal elements by d, This can also be seen in the augmented system (5). Where, $\bar{C}\bar{A}^r\bar{B} = 0$ for $r \in \{0, \ldots, d-1\}$. Consequently, the augmented system introduces md additional state variables. hence, It is required to have $n+md-p$ number of stable zero to achieve monotonicity and it has only $n-p$ stable zeros. Define the forward-shift matrix $Q(z) = z^d I_p$. Since $Q(z)$ is a diagonal matrix with identical diagonal entries, it commutes with $L(z)$, yielding, $Q(z)L(z) = L(z)Q(z)$. The modified system becomes:

$$G_m(z) = L(z)\left[Q(z)M(z)D(z)\right]R(z) \tag{11}$$

Pre multiplication with $Q(z)$ populates the transfer function with md stable zeros at the origin. Since the origin lies strictly inside the unit circle in discrete time, the introduced zeros at $z = 0$ are stable transmission zeros. In the time domain, pre-multiplication by $Q(z)$ corresponds to a forward shift of d steps in each output. now this modified system has $n+md-p$ stable zeros. Incorporating the exosystem dynamics $w(k+1) = Sw(k)$, the modified output error is:

$$e_m(k) = e(k+d) = \bar{C}\bar{A}^d X(k) + Du(k) + FS^d w(k) \tag{12}$$

Defining $C_m = \bar{C}\bar{A}^d$ and $F_m = FS^d$, the modified LTI system is:

$$\begin{cases} X(k+1) = \bar{A}X(k) + \bar{B}u(k) + Hw(k) \\ e_m(k) = C_m X(k) + Du(k) + F_m w(k) \end{cases} \tag{13}$$

Multiplication by z^d introduces zeros at $z = 0$ with multiplicities $m_a(0) = rd$ and $m_g(0) = r$, where $r = \operatorname{rank}(G(z))$. The modified output matrix C_m induces the required md stable zeros at the origin, enabling a state-feedback gain K that guarantees monotonicity after d steps.

[6] gives the method for monotonic output regulation with disturbance rejection. Let exogenous signal is generated from autonomous LTI system (2). Which consist of trajectory to be followed by the LTI system and disturbance

to be rejected. In[6] it is given that input of the form

$$u(k) = KX(k) + K_1 w(k) \tag{14}$$

solves the output regulation problem where K is any matrix which stabilizes the closed loop $(\bar{A} + \bar{B}K)$ and all output tracking error converges to zero monotonically,
$K_1 = \bar{\Gamma} + K\bar{\Pi}$
Where $\bar{\Gamma}$ and $\bar{\Pi}$ are the solution of

$$\begin{cases} \bar{\Pi}S = \bar{A}\bar{\Pi} + \bar{B}\bar{\Gamma} + \bar{H} \\ 0 = C_m\bar{\Pi} + D\bar{\Gamma} + F_m \end{cases} \tag{15}$$

This establishes the feasibility of achieving monotonic output regulation for LTI system with a fixed uniform input delay d while rejecting disturbances. The following theorem from [6] gives the observer design stated as below.

Theorem 2. *([6],[11]) Assume system (13) satisfies following assumptions:*

- *The matrix S has no eigenvalues greater than one.*
- *The pair $(\bar{A}, \bar{B})$ is stabilizable.*
- *The pair*
$$\left(\begin{bmatrix} C_m & F_m \end{bmatrix}, \begin{bmatrix} \bar{A} & \bar{H} \\ 0 & S \end{bmatrix}\right)$$
is detectable.

then, exact output regulation problem is solvable by measurement feedback if and only if there exist matrices $\bar{\Gamma}$ and $\bar{\Pi}$ satisfying (15). A suitable measurement feedback controller is then given by

$$\Sigma_c = \begin{cases} \begin{bmatrix} \hat{X}(k+1) \\ \hat{w}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{H} \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{X}(k) \\ \hat{w}(k) \end{bmatrix} + \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} u(k) \\ \qquad + \begin{bmatrix} K_A \\ K_S \end{bmatrix} \left(\begin{bmatrix} C_m & D & F_m \end{bmatrix} \begin{bmatrix} \hat{X}(k) \\ u(k) \\ \hat{w}(k) \end{bmatrix} - e_m(k) \right) \end{cases}$$

$$u(k) = K\hat{X}(k) + (\bar{\Gamma} - K\bar{\Pi})\hat{w}(k)$$

where, $\hat{X}(k)$ and $\hat{w}(k)$ are the estimates of $X(k)$ and $w(k)$ respectively and K, K_A, K_S are chosen such that

$$\bar{A} + \bar{B}K \quad \text{and} \quad \begin{bmatrix} \bar{A} + K_A C_m & \bar{H} + K_A F_m \\ K_S C_m & S + K_S F_m \end{bmatrix}$$

are both schur stable.

V. Numerical Examples

Example 1: Consider a spring-mass system where forces f_1, f_2, and f_3 act on masses M_1, M_2, and M_3, respectively, where the corresponding masses are m_1, m_2, and m_3. The parameters k_1, k_2, and k_3 denote the spring constants, d_1,d_2,d_3 denotes the damping constant and y_1, y_2, and y_3 represent the positions of the masses measured from the wall. p_1,p_2 and p_3 are the disturbance caused by the external linear autonomous system.

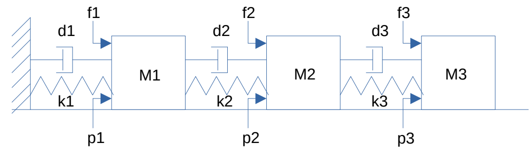

Fig. 2. Spring–mass system

State vector:

$$x = \begin{bmatrix} y_1 & y_2 & y_3 & \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{bmatrix}^T$$

The system can be represented in state-space form as:

$$\begin{aligned} \dot{x(t)} &= Ax(t) + Bu(t) + H_c w(t) \\ y(t) &= Cx(t) \\ e(t) &= Cx(t) + Fw(t) \\ \dot{w(t)} &= S_c w(t) \end{aligned}$$

Continuous-time system matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & -\frac{d_1+d_2}{m_1} & \frac{d_2}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{k_3}{m_2} & \frac{d_2}{m_2} & -\frac{d_2+d_3}{m_2} & \frac{d_3}{m_2} \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} & 0 & \frac{d_3}{m_3} & -\frac{d_3}{m_3} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_c = \begin{bmatrix} 0 & -1.5 & 0 & 0 & 0 \\ 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} F = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Discretized system (sampling time $T_s = 0.01$ s):
The system can be represented in state-space form as:

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) + Hw(k) \\ y(k) &= Cx(k) \\ e(k) &= Cx(k) + Fw(k) \\ w(k) &= Sw(k+1) \end{aligned}$$

$k_1 = 100N/m$, $k_2 = 150N/m$, $k_3 = 200N/m$,
$m_1 = 30kg$, $m_2 = 20kg$, $m_3 = 10kg$,

$d_1 = 10N\cdot s/m,\ d_2 = 10N\cdot s/m,\ d_3 = 10N\cdot s/m$

$$A_d = \begin{bmatrix} 0.9996 & 0.0002 & 0 & 0.01 & 0 & 0 \\ 0.0004 & 0.9991 & 0.0005 & 0 & 0.0099 & 0 \\ 0 & 0.001 & 0.999 & 0 & 0.0001 & 0.0099 \\ -0.0829 & 0.0495 & 0.0002 & 0.9929 & 0.0036 & 0 \\ 0.0744 & -0.1734 & 0.0989 & 0.0053 & 0.9892 & 0.0054 \\ 0.0004 & 0.198 & -0.1984 & 0 & 0.0109 & 0.9891 \end{bmatrix}$$

$$B_d = 10^{-3}\begin{bmatrix} 0.0017 & 0 & 0 \\ 0 & 0.0025 & 0 \\ 0 & 0 & 0.005 \\ 0.3322 & 0.0009 & 0 \\ 0.0009 & 0.4974 & 0.0026 \\ 0 & 0.0026 & 0.9947 \end{bmatrix}$$

$$H = 10^{-3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 10 & -0.1 & 0 & 0 & 0 \\ 0.1 & 9.9 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1.00 & -0.02 & 0 & 0 & 0 \\ 0.02 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 1.00 & -0.05 & 0 \\ 0 & 0 & 0.05 & 1.00 & 0 \\ 0 & 0 & 0 & 0 & 0.98 \end{bmatrix}$$

Zeros of the system: $\{1, 1, 1\}$
Since the system has three zeros at one with geometric multiplicity 3, it is possible to construct a state feedback matrix K.

$$K = \begin{bmatrix} 250.00 & -150.00 & 0.00 & -8.85 & -10.65 & 0.00 \\ -150.00 & 350.00 & -200.00 & -10.70 & -18.46 & -10.84 \\ 0.00 & -200.00 & 200.00 & 0.00 & -10.90 & -19.16 \end{bmatrix}$$

Eigen value of $(A_d + B_d K)$ are $\{1,\ 1,\ 1,\ 0.97,\ 0.98,\ 0.99\}$,
Observer gain:

$$K_A = \begin{bmatrix} 4.20 & -9.28 & -6.23 \\ 11.89 & -10.26 & -14.08 \\ 12.35 & -10.15 & -13.90 \\ -8.68 & -2.73 & 0.05 \\ 2.05 & -1.70 & -6.70 \\ 3.57 & 0.26 & -2.34 \end{bmatrix}$$

$$K_S = \begin{bmatrix} -55.68 & -22.73 & 55.89 \\ 13.41 & -29.43 & -64.67 \\ -8.05 & -2.69 & 0.03 \\ 2.14 & -1.23 & -6.74 \\ 3.63 & 0.34 & -1.74 \end{bmatrix}$$

Initial state of the LTI system and exosystem:

$x(0) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}^T$,
$w(0) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$

$$K_1 = \begin{bmatrix} -29.9987 & 0.2253 & 26.0789 & -150.5577 & 0.0001 \\ -0.1503 & -19.9994 & 100.4429 & 37.9628 & -0.0546 \\ 0.0001 & -0.0000 & -0.2675 & -0.1006 & 10.2532 \end{bmatrix}$$

Control input $u(k) = Kx(k) + K_1 w(k)$ yields the desired transient performance. Fig. 3 and 4 show the tracking error and velocity of each mass, respectively.

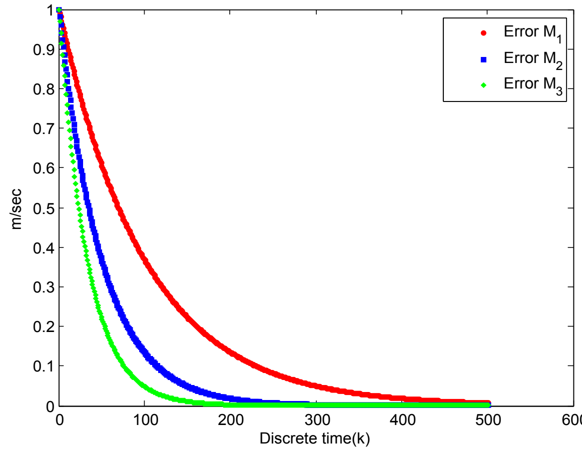

Fig. 3. Tracking error of each mass

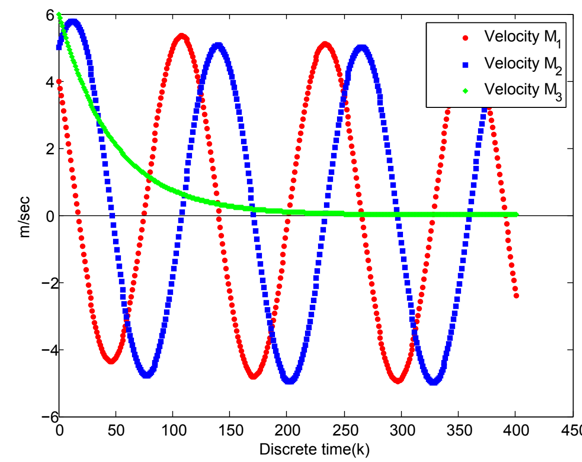

Fig. 4. Velocity of each mass

Example 2: Consider the same system with an input delay of $4T$. The 18-state augmented system is represented as:

$$\begin{aligned} X(k+1) &= \bar{A}X(k) + \bar{B}u(k), \\ y(k) &= \bar{C}X(k), \\ e(k) &= \bar{C}X(k) + Fw(k). \end{aligned}$$

where $X(k) = [x^\top(k), u^\top(k-3), \ldots, u^\top(k-1)]^\top$, $\bar{C} = [C\ \mathbf{0}]$, and the state transition matrix structures are standard delay-line forms. The system has a relative degree of 4. The modified output matrix is $C_m = \bar{C}\bar{A}^4$, and $F_m = FS^4$ yielding a modified regulated error $e_m = C_m X(k) + F_m w(k)$. The open-loop spectrum with modified output matrix contains zeros at $z = 0$ ($m_g(0) = 3, m_a(0) = 12$) and $z = 1$ ($m_g(1) = 3, m_a(1) = 3$). Since the system possesses 15 stable zeros, a state-feedback matrix can be designed to guarantee monotonicity.

Given initial states $x(0) = [1\ 2\ 3\ 4\ 5\ 6]^\top$ and zero past inputs $u(-4) = \cdots = u(-1) = \mathbf{0}_{3\times 1}$, the control law (14) can be structured as $u(k) = K_{11}\hat{x}(k) + K_{12}u_{past}(k) + K_1\hat{w}(k)$, where $u_{past}(k) = [u^\top(k-4) \ldots u^\top(k-1)]^\top$. The computed gains are:

$$K_{11}^{\top} = \begin{bmatrix} 247.29 & -148.91 & -4.44 \\ -141.53 & 343.60 & -201.47 \\ -5.23 & -192.87 & 205.94 \\ 0.95 & -16.56 & -0.38 \\ -16.02 & -4.84 & -18.97 \\ -0.36 & -18.38 & -10.71 \end{bmatrix}$$

$$K_{12}^{\top} = 10^{-2} \times \begin{bmatrix} -0.0 & -1.0 & -0.0 \\ -1.0 & -0.0 & -1.0 \\ -0.0 & -2.0 & -1.0 \\ -0.0 & -0.0 & -0.0 \\ -1.0 & -0.0 & -1.0 \\ -0.0 & -2.0 & -2.0 \\ -0.0 & -0.0 & -0.0 \\ -1.0 & -1.0 & -1.0 \\ -0.0 & -1.0 & -2.0 \\ -0.0 & -0.0 & -0.0 \\ -1.0 & -1.0 & -1.0 \\ -0.0 & -2.0 & -2.0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -29.29 & 6.84 & -450.60 & -394.49 & -0.00 \\ -5.50 & -19.80 & 279.18 & -285.76 & 0.18 \\ -0.06 & -0.60 & -0.74 & 0.76 & -34.23 \end{bmatrix}$$

$$K_S = \begin{bmatrix} -34.00 & 42.65 & -8.96 \\ -25.43 & -42.70 & -8.79 \\ -0.79 & 1.61 & -0.24 \\ -1.14 & -0.86 & -0.15 \\ -0.03 & -0.32 & 0.71 \end{bmatrix}$$

The transposed observer gain matrix is given compactly by:

$$K_A^{\top} = \begin{bmatrix} -5.94 & -4.66 & -4.70 & -2.87 & -0.96 & -0.05 & \mathbf{0}_{1\times 12} \\ -10.86 & -16.76 & -17.07 & 1.78 & -3.65 & -0.47 & \mathbf{0}_{1\times 12} \\ -1.00 & 0.04 & 0.92 & -0.65 & -0.22 & 0.78 & \mathbf{0}_{1\times 12} \end{bmatrix}$$

Fig. 5 and 6, shows the tracking errors and velocities of each mass respectively. Tracking error converge to zero monotonically after the required four-step delay.

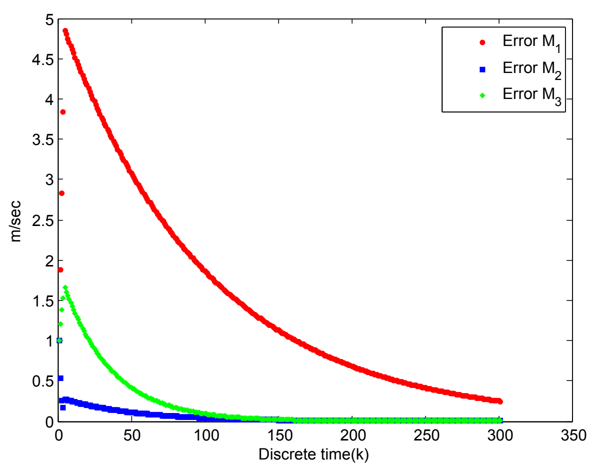

Fig. 5. Tracking error of each mass

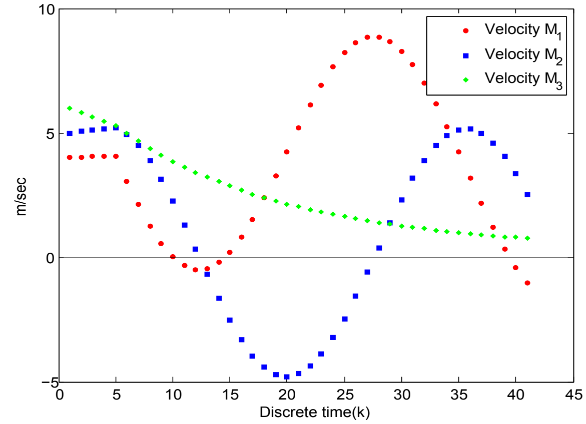

Fig. 6. Velocity of each mass

VI. Conclusion and Future Scope

This work establishes the feasibility conditions for achieving a monotonic output regulation in discrete MIMO LTI systems characterized by repeated zeros at one and having uniform input delays. It is also verified on discretized spring mass system having 3 zeros at one and uniform input delay. Future work will extend this feasibility result for remaining zeros on the unit circle. Additionally, since the present augmented approach leads to a high-dimensional system and restricts delays to multiples of the sampling time T, further research will aim to develop alternative methods capable of handling arbitrary delays more efficiently.

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