An Exponentially Convergent Velocity Observer for Nonholonomic Mobile Robots with Nonlinear Friction Terms

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***Abstract*— Friction is a natural phenomenon in the motion of mechanical systems, which could be complicated to compensate for by a control law if it is not static. In this note, we proposed a globally convergent velocity observer for mobile robots modeled with static and Coulomb frictions. Since Coulomb friction is described by a nonlinear term where the velocity state is its argument, we deal with this challenging problem by using the immersion and invariance technique together with a monotonicity property. To test the robustness of the proposed velocity observer and its use for regulation of all states with an output feedback controller, a simulation is conducted.**

I. INTRODUCTION

Wheeled mobile robots have been extensively adopted in many practical applications, including robotics, medical services, logistics, forestry, mining, and defense activities. These vehicles are used for numerous tasks such as rescue operations, industrial automation, warehouse management, inspection of hazardous environments, and agricultural operations. Compared to primarily fixed robotic systems, such as robot manipulators, wheeled robotic platforms provide efficient mobility and extended operational range, highlighting their versatility. The dynamic model of mobile robots is relatively straightforward to obtain, and classical formalisms such as the Euler–Lagrange or Port-Hamiltonian approaches are commonly used. Moreover, the intrinsic nonholonomic constraints of these systems allow for a reduction of the dynamic model. However, this same property complicates the design of stabilizing controllers, since Brockett's theorem states that nonholonomic systems cannot be stabilized using smooth static state-feedback control laws [1].

When dynamic models are constructed to describe mobile robots, several physical phenomena should be considered, such as viscous friction, Coulomb friction, LuGre friction, skidding and longitudinal slipping effects on the wheels, among others [2], [3]. Added to these effects, it should be noted that in general, the velocity state in physical systems, including nonholonomic robots, is not measurable. This presents a challenging problem when designing time-(varying) invariant controllers to stabilize nonholonomic robots since they typically need to know the system's position and velocity, whether it's in regulation or tracking tasks.

To measure velocity, expensive sensing systems such as motion capture systems [4] and vision-based localization systems [5] are sometimes employed, from which numerical velocity estimates can be obtained. However, when these estimates are corrupted by noise, the controller may destabilize the system. Furthermore, a rigorous stability analysis cannot generally be guaranteed when such signals are directly used in feedback control. For this reason, the design of velocity observers has become a common alternative. Nevertheless, most velocity observers reported in the literature do not account for nonlinear friction, slip, and skid effects in the wheels. The main reason is that incorporating these effects significantly increases the complexity of the observer design. Moreover, the literature contains only a limited number of velocity observers with exponential convergence for mobile robots with nonholonomic constraints.

In [6], the first global exponentially convergent velocity observer for those systems was established. The result is based on a change of coordinates on the momenta and the immersion and invariance (I&I) technique, which is a constructive methodology to design adaptive controllers and state observers for dynamical systems [7]. It is important to highlight that to deal with the Coriolis forces, a novel dynamic scaling approach was presented in that work. See also [8], where the dynamic scaling idea is exploited to solve the trajectory tracking problem with only position feedback in mechanical systems. A similar transformation to remove the quadratic terms in velocity was proposed in [9] and [10], so that a time varying output feedback controller using a passive velocity observer was presented for surface ships and mobile robots. Reference [11] utilizes the dynamic scaling technique introduced in [6] together with a change in coordinates to transform a constrained system into an integral cascade Euler-Lagrange form. Thus, the gyroscopic forces are compensated and a low-dimensional, exponentially convergent velocity observer is presented.

Even though the presence of quadratic terms in velocity is the main issue when designing velocity observers for nonholonomic robots, one cannot lose sight of factors such as unmodeled parameters, external disturbances or physical phenomena that appear in the modeling of real mobile robots, such as different frictions and skid and slip effects. The addition of these terms leads to a more complicated problem when trying to build velocity observers. For example, discontinuous and high gains adaptive velocity observers have been proposed in the literature [12]–[14] to handle external disturbances and unmodeled terms. The I&I theory– which differs from those techniques.– was used to design a simple continuously adaptive velocity observer for perturbed nonholonomic mobile robots [15]. Also, see [16].

The velocity estimation problem could be difficult to handle if *nonlinear* physical effects like friction, skidding, and longitudinal slipping are taken into account in mobile robots.

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These terms are particularly evident during the interaction of the tires with the road [17]. Thus, very few works have included those terms in the velocity observer design. Considering viscous friction terms, [18] proposed an adaptive velocity observer following the methodology of [19], where sufficient and necessary conditions were established to deal with the problem of having products between unknown parameters and non-measurable states. Based on learning techniques, such as machine learning [20] and fuzzy logic [21], the adaptive observer problem for path tracking of a class of wheeled mobile robots has been solved, where unmeasurable velocities and unknown skidding and slipping effects appear coupled in the robot dynamic. However, those techniques lack a formal stability analysis.

In addition, nonlinear friction in nonholonomic robots can lead to more complicated control and velocity estimation problems, as discussed in [22], [23]. This is because the unmeasurable velocity appears in a nonlinear function, such as a relay or hiperbolic tangent. In fact, in the literature, this friction type in nonholonomic mobile robots has not been considered when designing the velocity observer. Some works for stabilization of mechanical systems have been concentrated on compensating the unknown constant friction coefficient inherent to the friction modeling [24], [25]. Recently, in [26], based on the I&I methodology, a *continuous adaptive velocity observer* for simple mechanical systems with static and coulomb friction was developed for the first time.

From the above discussion, it is clear that the design of velocity observers for nonholonomic mobile robots with nonlinear friction remains an open problem. Thus, in this work, we propose a globally convergent velocity observer for those systems assuming that the constant friction coefficients are known. The design does not rely on the use of switching techniques, nor does it assume the boundedness of velocity signals at the beginning. In contrast, inspired by [26], the stability analysis is established by invoking the well-known monotonicity property. To the best of the author's knowledge, this is the first result estimating velocity in nonholonomic mobile robots with viscous and Coulomb friction, which is described by a nonlinear function.

The rest of the note is organized as follows: Section II presents the dynamic model of the nonholonomic robot; Section III contains our main result; and, finally, the simulations and conclusions are shown in Sections IV and V, respectively.

II. Mathematical Model

In this work we consider a differential wheeled mobile robot, whose geometrical center and center of mass are located at the same point with $z = \mathrm{col}(x, y) \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ defining its position and orientation, respectively. Hence, the robot dynamics, taking into account both viscous and Coulomb friction, is given by

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} v, \quad \dot{\theta} = \omega, \\ \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} &= M^{-1}\mathcal{B}\tau - \begin{bmatrix} f_{v_1} v + f_{v_2}\tanh(\alpha_1 v) \\ f_{\omega_1}\omega + f_{\omega_2}\tanh(\alpha_2\omega) \end{bmatrix}, \end{aligned} \tag{1}$$

where v and ω are the linear and the angular velocities; $M := \mathrm{diag}(m, I) \in \mathbb{R}^2$ is the inertia matrix with I being the moment of inertia and m being the mass. $f_{v_i} > 0$ and $f_{\omega_i} > 0$ are constant friction coefficients with $i = 1:2$ and α_1, α_2 are sufficiently large numbers, so that the $\tanh(\cdot)$ function qualify as a suitable smooth approximation of a relay, which describes the Coulomb friction. The input matrix $\mathcal{B}$ is given by

$$\mathcal{B} = \frac{1}{r}\begin{bmatrix} 1 & 1 \\ 2R & -2R \end{bmatrix} \tag{2}$$

with R being the distance between z and the wheels which have radius r; the control signal of the wheels is $\tau \in \mathbb{R}^2$. Making straightforward calculations, the model (1) can be rewritten in linear and angular dynamics as follows

$$\begin{aligned} \dot{\theta} =& \quad \omega \\ \dot{\omega} =& \quad u_2 - f_{\omega_1}\omega - f_{\omega_2}\tanh(\alpha_2\omega) \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{z} =& \quad h(\theta)v \\ \dot{v} =& \quad u_1 - f_{v_1}v - f_{v_2}\tanh(\alpha_1 v), \end{aligned} \tag{4}$$

where $\varphi(\theta) = \mathrm{col}(\cos(\theta), \sin(\theta)) \in \mathbb{R}^2$; u_1 and u_2 are the control-input signals, given by

$$u_1 = \frac{1}{mr}(\tau_1 + \tau_2), \quad u_2 = \frac{2R}{Ir}(\tau_1 - \tau_2).$$

Thus, the main objective of this work is to design an exponentially convergent observer for both linear and angular velocities, even when there is nonlinear Coulomb friction. Instrumental for the design of the velocity observer, the following assumption is used.

Assumption 1: The orientation θ and the position z of the robot are available for measurement. Moreover, the friction coefficients f_{v_1}, f_{v_2}, f_{ω_1} and f_{ω_2} are known. ◁

III. Main Result

The proposed velocity observer is based on the Immersion and Invariance (I&I) methodology proposed in [7]. First, estimation errors are defined as follows

$$\bar{v} = \hat{v} - v, \quad \bar{\omega} = \hat{\omega} - \omega, \quad \bar{\theta} = \hat{\theta} - \theta \tag{5}$$

where $\hat{\omega}$, $\hat{v}$, $\hat{\theta}$ are the estimators for the linear and the angular velocities and the orientation, respectively. According to the I&I methodology, the observer consists of two terms—an integral and a proportional. Hence, the estimators are defined as

$$\hat{v} = v_I + v_P(\hat{\theta}, z), \quad \hat{\omega} = \omega_I + \omega_P(\theta). \tag{6}$$

Under Assumption 1, the main objective is to determine the dynamics of the integral terms v_I, ω_I and to define the proportional components v_P, ω_P, such that the estimation errors in (5) converge exponentially to zero for all initial conditions.

The adaptive velocity observer proposed in this work exploits a monotonicity property, which is defined as follows.

Lemma 1: A mapping $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ is *strongly monotone* if there exists a constant $c > 0$ such that

$$(a-b)[\mathcal{L}(a) - \mathcal{L}(b)] \geq c(a-b)^2 > 0, \quad \forall a, b \in \mathbb{R},\ a \neq b. \tag{7}$$

At this point, we are ready to present our main result.

Proposition 1: Consider the models given by (3) and (4). Define the integral dynamics of (6) and the $\hat{\theta}$-dynamics as follows

$$\begin{aligned} \dot{v}_I &= -a_3\varphi^{\perp}(\hat{\theta})z\dot{\hat{\theta}} - a_3\varphi^{\top}(\hat{\theta})\varphi(\theta)\hat{v} + u_1 \\ &\quad -f_{v_1}\hat{v} - f_{v_2}\tanh(\alpha_1\hat{v}), && (8)\\ \dot{\omega}_I &= -a_1\hat{\omega} + u_2 - f_{\omega_1}\hat{\omega} - f_{\omega_2}\tanh(\alpha_2\hat{\omega}), && (9)\\ \dot{\hat{\theta}} &= \hat{\omega} - k_a(r_v)\bar{\theta}, && (10) \end{aligned}$$

where r_v is a dynamic scaling to be defined latter and $\varphi^{\perp}(\hat{\theta})$ verifying $\varphi^{\perp}(\cdot)\varphi(\cdot) = 0$. Moreover, the proportional terms have been set to

$$\omega_P = a_1\theta, \quad v_P = a_3 z^{\top}\varphi(\hat{\theta}), \tag{11}$$

with positive proportional gains a_1 and a_3 and

$$k_a(r_v) = \lambda_5 + \frac{1}{2} + 2a_3\bar{r}_v r_v, \tag{12}$$

where $\lambda_5 > 0$ and $\bar{r}_v = r_v - 1$. Then, for all initial conditions $(z(0), \theta(0), v(0), \omega(0)) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we ensure that

$$\left|\left| \begin{bmatrix} \bar{v}(t) \\ \bar{\omega}(t) \end{bmatrix} \right|\right| \leq m \exp^{-\rho t} \left|\left| \begin{bmatrix} \bar{v}(0) \\ \bar{\omega}(0) \end{bmatrix} \right|\right|, \quad \forall t \geq 0, \tag{13}$$

for some positive m and ρ. ◁

Proof: On one hand, the dynamic behavior of $\bar{\omega}$ is given by

$$\dot{\bar{\omega}} = \dot{\omega}_I + a_1(\hat{\omega} - \bar{\omega}) - u_2 + f_{\omega_1}(\hat{\omega} - \bar{\omega}) + \tanh(\alpha_2\omega).$$

Choosing $\dot{\omega}_I$ as in (9), we get

$$\dot{\bar{\omega}} = -a_1\bar{\omega} - f_{\omega_1}\bar{\omega} - f_{\omega_2}[\tanh(\alpha_2\hat{\omega}) - \tanh(\alpha_2\omega)]. \tag{14}$$

Now, consider the Lyapunov function

$$\bar{H}_\omega = \frac{1}{2}\alpha_2\bar{\omega}^2. \tag{15}$$

Taking its time derivative along the solutions (14), yields,

$$\dot{\bar{H}}_\omega = -\alpha_2(a_1 + f_{\omega_1})\bar{\omega}^2 - f_{\omega_2}\alpha_2\bar{\omega}[\tanh(\alpha_2\hat{\omega}) - \tanh(\alpha_2\omega)]. \tag{16}$$

Invoking Lemma 1 we obtain

$$\begin{aligned} \dot{\bar{H}}_\omega &\leq -\alpha_2(a_1 + f_{\omega_1} + \alpha_2 f_{\omega_2})\bar{\omega}^2 \\ &\leq -2\lambda_1\bar{H}_\omega \end{aligned} \tag{17}$$

with positive $\lambda_1 = a_1 + f_{\omega_1} + \alpha_2 f_{\omega_2}$.

Clearly, from (15) and (17) we prove that $\bar{\omega} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and consequently that $\bar{\omega}$ converges globally exponentially to zero.

On the other hand, computing the time derivative of $\bar{v}$, we get

$$\begin{aligned} \dot{\bar{v}} &= \dot{v}_I + a_3(z^{\top}\nabla\varphi(\hat{\theta})\dot{\hat{\theta}} + \varphi^{\top}(\hat{\theta})\dot{z}) - \dot{v} \\ &= \dot{v}_I + a_3\varphi^{\perp}(\hat{\theta})z\dot{\hat{\theta}} + a_3\varphi^{\top}(\hat{\theta})\varphi(\theta)[\hat{v} - \bar{v}] \\ &\quad - u_1 + f_{v_1}(\hat{v} - \bar{v}) + f_{v_2}\tanh(\alpha_1 v). \end{aligned} \tag{18}$$

Selecting $\dot{v}_I$ as in (8), yields

$$\begin{aligned} \dot{\bar{v}} &= -a_3\varphi^{\top}(\hat{\theta})\varphi(\theta)\bar{v} - f_{v_1}\bar{v} \\ &\quad - f_{v_2}[\tanh(\alpha_1\hat{v}) - \tanh(\alpha_1 v)], \\ &= -a_3[\pm\varphi^{\top}(\theta) - \varphi^{\top}(\hat{\theta})]\varphi(\theta)\bar{v} - f_{v_1}\bar{v} \\ &\quad - f_{v_2}[\tanh(\alpha_1\hat{v}) - \tanh(\alpha_1 v)], \\ &= -a_3\bar{v} + \Delta_v(\hat{\theta}, \theta)\bar{v} - f_{v_1}\bar{v} \\ &\quad - f_{v_2}[\tanh(\alpha_1\hat{v}) - \tanh(\alpha_1 v)], \end{aligned} \tag{19}$$

with

$$\Delta_v(\hat{\theta}, \theta) := -a_3[\varphi^{\top}(\hat{\theta}) - \varphi^{\top}(\theta)]\varphi(\theta).$$

It is noteworthy to mention that $\Delta_v = 0$, if $\bar{\theta} = 0$, and

$$||\Delta_v(\hat{\theta}, \theta)|| \leq 2a_3\bar{\theta}. \tag{20}$$

Thus, Δ_v is a vanishing term that will be dominated by using the dynamic scaling factor and the $\hat{\theta}$ dynamics.

Defining

$$z_v = \frac{1}{r_v}\bar{v}, \tag{21}$$

with r_v being a *dynamic scaling* factor, with dynamics defined as

$$\dot{r}_v := -\frac{a_3}{4}\bar{r}_v + 2a_3 r_v\bar{\theta}^2, \quad r_v(0) > 1, \tag{22}$$

where $\bar{r}_v(t)$ is $\bar{r}_v(t) = r_v(t) - 1$, so that the set $\{r_v \in \mathbb{R} | r_v \geq 0\}$ is invariant for the dynamics (22).

Hence, the time derivative of (21) taken along the dynamics (19) and (22) has the following form

$$\begin{aligned} \dot{z}_v &= \frac{1}{r_v}\dot{\bar{v}} - \frac{\dot{r}_v}{r_v^2}\bar{v} \\ &= \frac{1}{r_v}\Big[-(a_3 + f_{v_1} - \Delta_v(\hat{\theta}, \theta))\bar{v} \\ &\quad - f_{v_2}(\tanh(\alpha_1\hat{v}) - \tanh(\alpha_1 v))\Big] - \frac{\dot{r}_v}{r_v}z_v \\ &= -(a_3 + f_{v_1} - \Delta_v(\hat{\theta}, \theta))z_v \\ &\quad - \frac{f_{v_2}}{r_v}\Big(\tanh(\alpha_1\hat{v}) - \tanh(\alpha_1 v)\Big) - \frac{\dot{r}_v}{r_v}z_v. \end{aligned} \tag{23}$$

Moreover, the $\bar{\theta}$ dynamics is given by

$$\dot{\bar{\theta}} = \dot{\hat{\theta}} - (\hat{\omega} - \bar{\omega}) := -k_a(r_v)\bar{\theta} + \bar{\omega}, \tag{24}$$

where we have chosen $\dot{\hat{\theta}}$ as in (10).

Finally, consider the Lyapunov function

$$H_z = \frac{1}{2}\left(\alpha_2\bar{\omega}^2 + \alpha_1 z_v^2 + \bar{\theta}^2 + \bar{r}_v^2\right). \tag{25}$$

Taking its time derivative along the dynamics (14), (23) and (24), we obtain

$$\begin{aligned}
\dot{H}_z &= \alpha_2\bar{\omega}\dot{\bar{\omega}} + \alpha_1 z_v \dot{z}_v + \bar{\theta}\dot{\bar{\theta}} + \bar{r}_v \dot{r}_v \\
&= \alpha_2\bar{\omega}\dot{\bar{\omega}} - \alpha_1 z_v \left(a_3 + f_{v_1} - \Delta_v(\hat{\theta},\theta) + \frac{\dot{r}_v}{r_v} \right) z_v \\
&\quad - \alpha_1 f_{v_2} \frac{z_v}{r_v} [\tanh(\alpha_1 \hat{v}) - \tanh(\alpha_1 v)] \\
&\quad - k_a(r_v)\bar{\theta}^2 + \bar{\theta}\bar{\omega} + \dot{\bar{r}}_v \bar{r}_v \\
&= \alpha_2\bar{\omega}\dot{\bar{\omega}} - \alpha_1 \left(a_3 + f_{v_1} - \Delta_v(\hat{\theta},\theta) + \frac{\dot{r}_v}{r_v} \right) z_v^2 \\
&\quad - f_{v_2} \frac{\alpha_1 \bar{v}}{r_v^2} [\tanh(\alpha_1 \hat{v}) - \tanh(\alpha_1 v)] \\
&\quad - k_a(r_v)\bar{\theta}^2 + \bar{\theta}\bar{\omega} + \dot{\bar{r}}_v \bar{r}_v. \qquad (26)
\end{aligned}$$

Now, by using (17) and Lemma 1 we get

$$\begin{aligned}
&\le -\alpha_2\lambda_1\bar{\omega}^2 - \alpha_1 \left(f_{v_1} + \frac{a_3}{2} - \frac{1}{2a_3}||\Delta_v||^2 + \frac{\dot{r}_v}{r_v} \right) z_v^2 \\
&\quad - k_a\bar{\theta}^2 + \frac{1}{2}\bar{\theta}^2 + \frac{1}{2}\bar{\omega}^2 + \dot{r}_v\bar{r}_v - f_{v_2}\frac{\alpha_1^2}{r_v^2}\bar{v}^2, \qquad (27)
\end{aligned}$$

where $||\cdot||$ is the matrix induced 2-norm and used the Young's inequality. Using (20)–(22), yields

$$\begin{aligned}
\dot{H}_z \le &-\alpha_1 \left(f_{v_1} + f_{v_2}\alpha_1 + a_3 \left(\frac{1}{2} - \frac{\bar{r}_v}{4r_v} \right) \right) z_v^2 \\
&- \left(\alpha_2\lambda_1 - \frac{1}{2} \right)\bar{\omega}^2 - \left(k_a - \frac{1}{2} - 2a_3\bar{r}_v r_v \right)\bar{\theta}^2 - \frac{a_3}{4}\bar{r}_v^2. \qquad (28)
\end{aligned}$$

Invoking the fact that $0 < \frac{\bar{r}_v}{r_v} \le 1$ and since λ_1 contains the free gain a_1, we can define the positive gains

$$\lambda_2 = \alpha_1(f_{v_1} + f_{v_2}\alpha_1 + \frac{a_3}{4}), \quad \lambda_3 = \alpha_2\lambda_1 - \frac{1}{2}, \quad \lambda_4 = \frac{a_3}{4}$$

and $k_a(r_v)$ as (12), such that

$$\dot{H}_z \le -\lambda_2 z_v^2 - \lambda_3\bar{\omega}^2 - \lambda_4\bar{r}_v^2 - \lambda_5\bar{\theta}^2.$$

Thus, we ensure that $\bar{\omega}, z_v, \bar{\theta}, \bar{r}_v \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Furthermore, there exists ρ_1 such that $\dot{H}_z \le -\rho_1 H_z$, with $\rho > 0$, which ensures that they converge exponentially to zero. Finally, using (21) we ensures that $\bar{v}$ converges exponentially to zero. This completes the proof. ■

IV. Simulations

In this section, we present simulations to assess the proposed velocity observer. First we show its robustness in open loop and after we present its good performance in closed-loop using a novel output feedback scheme. The simulation parameters are $m = 5.64$ kg, $I = 3.115$ kg·m^2, $r = 0.09$ m and $R = 0.157$ m. Besides, we set $f_{v_1} = 0.15$, $f_{v_2} = 0.14$, $\alpha_1 = 120$, $f_{\omega_1} = 0.13$, $f_{\omega_2} = 0.15$ and $\alpha_2 = 110$.

A. Robustness

To evaluate the robustness of the velocity observer, we consider that the measurable signals z and θ are corrupted with the noise signal shown in Fig. 1. Thus, the simulation was carried out with the input signal

$$\tau = \mathcal{B}^{-1}M \begin{bmatrix} -k_x h^\top(\theta(t) + n(t))(z(t) + n(t)) \\ -k_c(\theta(t) + n(t)) \end{bmatrix}$$

with $k_x = 4$ and $k_c = 3.75$. The observer gains were set as $a_1 = 7$, $a_3 = 9$ and $\lambda = 4$. The initial conditions were selected as $\theta(0) = 0.5$, $z(0) = \mathrm{col}(0.1, 0.3)$, $\omega(0) = v(0) = 0$, $\omega_I(0) = v_I(0) = 0.5$, $\hat{\theta}(0) = 0.6$ and $r_v(0) = 3$. Both, initial conditions and observer gains were chosen arbitrarily.

Fig. 2 demonstrates the transient behavior of estimator errors, which supports the robustness of the proposed observer, as even with noise, the errors remain close to zero.

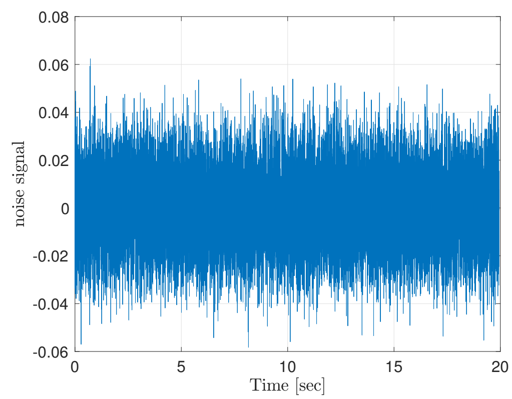

Fig. 1. Noise signal added to θ and z

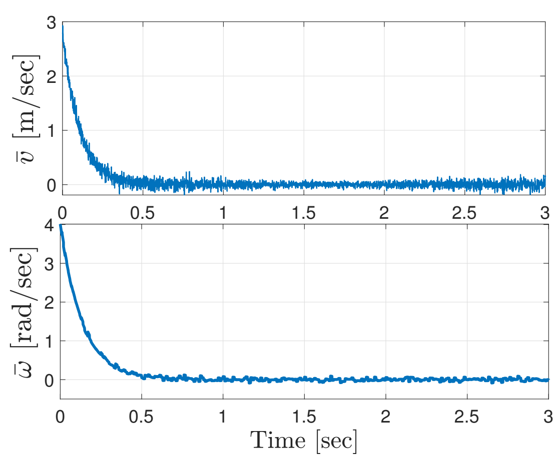

Fig. 2. Transient behavior of $\bar{v}$ and $\bar{\omega}$

B. Closed loop controller

Our second scenario involves designing an implementable control. Therefore, we first design an ideal controller, assuming that the velocities v and ω can be measured, thus $u = u^\star$

with

$$\begin{aligned} u_1^\star &= -p_v h(\theta)^\top \tilde{z} - d_v v + f_{v_1} v + f_{v_2} \tanh(\alpha_1 v), \\ u_2^\star &= -p_\omega \tilde{\theta} - d_\omega \omega + K_I f(t) h(\theta)^\perp \tilde{z} + f_{\omega_1} \omega \\ &\quad + f_{\omega_2} \tanh(\alpha_2 \omega), \end{aligned} \tag{29}$$

where $\tilde{z}(t) = z(t) - z_d$ and $\tilde{\theta}(t) = \theta(t) - \theta_d$, with $z_d \in \mathbb{R}^2$ and θ_d as the desired position and orientation, respectively and; control gains p_v, d_v, p_ω, d_ω and K_I. [1] We notice that the control law has a simple PD structure plus a time varying term $f(t)$ which verifies the Brockett's condition.

Now, we replace the estimated velocity designed in Proposition 1—in a certainty-equivalent manner—in (29). Therefore, the proposed output feedback controller has the following form

$$\begin{aligned} u_1 &= -p_v h(\theta)^\top \tilde{z} - d_v \hat{v} + f_{v_1} \hat{v} + f_{v_2} \tanh(\alpha_1 \hat{v}), \\ u_2 &= -p_\omega \tilde{\theta} - d_\omega \hat{\omega} + K_I f(t) \varphi(\theta)^\perp \tilde{z} + f_{\omega_1} \hat{\omega} \\ &\quad + f_{\omega_2} \tanh(\alpha_2 \hat{\omega}). \end{aligned} \tag{30}$$

Thus, the systems (3) and (4) in closed loop with (30) take the form

$$\begin{aligned} \dot{v} = &- p_v h(\theta)^\top \tilde{z} - d_v \bar{v} - d_v v + f_{v_1} \bar{v} \\ &+ f_{v_2}[\tanh(\alpha_1 \hat{v}) - \tanh(\alpha_1 v)] \\ \dot{\omega} = &- p_\omega \tilde{\theta} - d_\omega \bar{\omega} - d_\omega \bar{\omega} + K_I f(t) \varphi(\theta)^\perp \tilde{z} + f_{\omega_1} \bar{\omega} \\ &+ f_{\omega_2}[\tanh(\alpha_2 \hat{\omega}) - \tanh(\alpha_2 \omega)]. \end{aligned} \tag{31}$$

Clearly, to obtain the closed-loop we have replaced in the controller the estimated quantities $\hat{(\cdot)}$ by $\tilde{(\cdot)} + (\cdot)$. Due to space limitations, the stability proof of the proposed output feedback controller for nonholonomic mobile robots with nonlinear friction terms is omitted and will be reported elsewhere.

Figures 3 and 4 illustrate the convergence of the coordinate errors to zero for the following initial conditions: $z(0) = (0.1, 0.2)m$ and $\theta(0) = 0.5rad$ and the desired point $z_d = (4, 3)m$ and $\theta_d = -\frac{\pi}{6}$.

Figure 5 shows the transient behavior of the estimator errors for both velocities, while Fig. 6 shows the estimator error for orientation and dynamic $\bar{r}_v$. Similarly to the open-loop case, the estimates are ensured. Finally, the input control appears in Fig. 7.

V. Conclusions

We presented a solution for designing a globally convergent velocity observer for nonholononic mobile robots with linear and nonlinear friction. The observer was created using the well-known I&I methodology and exploiting a monotonicity principle as [26]. This contrasts with conventional discontinuous techniques, which are normally used for friction compensation in simple mechanical systems. The simulation results demonstrate the robustness of the proposed velocity observer, taking into account the presence of additive noise in the measurable states.

[1]This ideal controller, with all friction terms equal to zero in both model (1) and control (29) was reported in [27] for the multiagents case, ensuring consensus of both position and angular states.

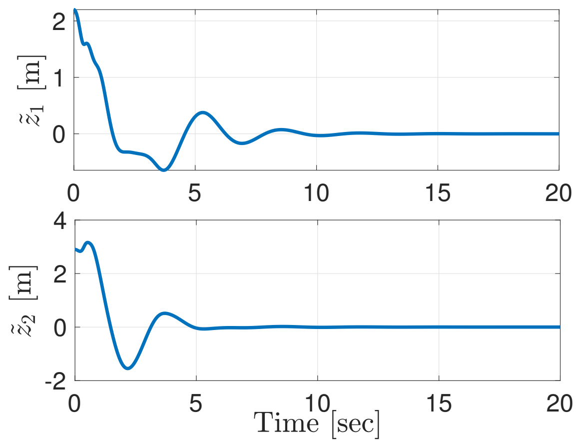

Fig. 3. Transient behavior of the position errors $\tilde{z}$

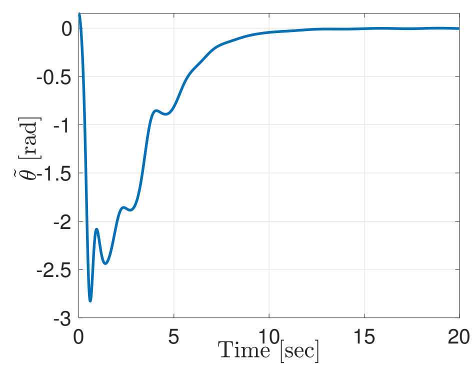

Fig. 4. Transient behavior of the angular error $\tilde{\theta}$

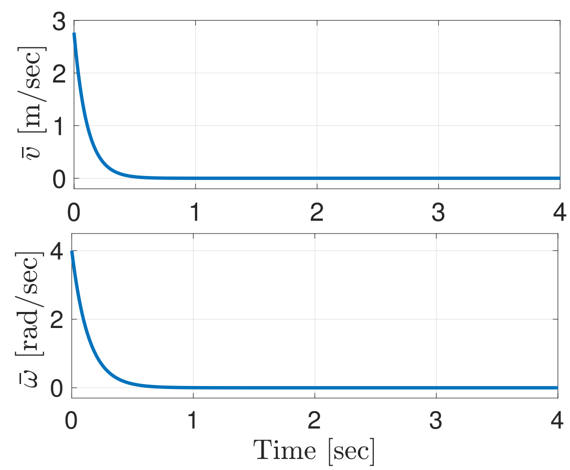

Fig. 5. Transient behavior of $\bar{v}$ and $\bar{\omega}$

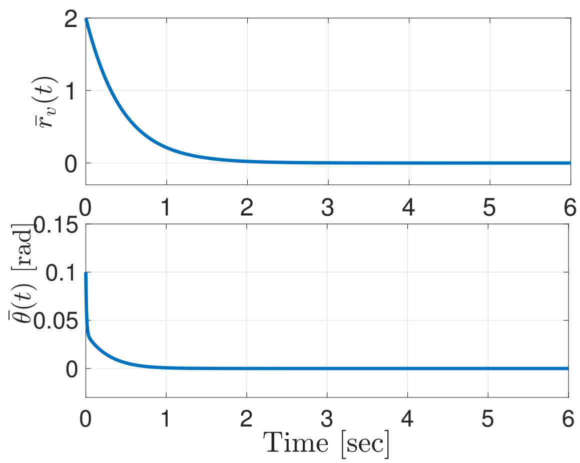

Fig. 6. Transient behavior of $\bar{\theta}1$ and $\bar{r}_v$

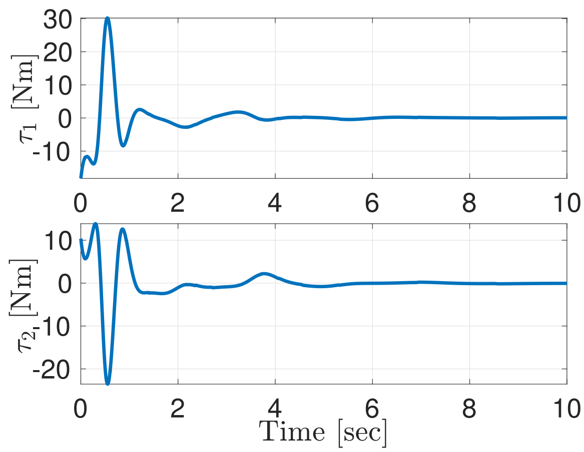

Fig. 7. Transient behavior of the input controller τ

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