On a Problem Related to Achieving a Given Level by a Random Process

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***Abstract*— The problem of the first achievement of a given level by a random process is considered. The previously obtained general results are specified for a smooth Gaussian process.**

I. INTRODUCTION

We consider the problem of finding the probability that the first exit of an n-dimensional random process $\mathbf{y}(x)$ to the boundary ∂Q of a given domain $Q \subset \mathbb{R}^n$ will occur at some moment $\tilde{x}$ from a given interval (x', x'') and this exit will take place on a given part of boundary ∂Q. Here $\mathbb{R}^n$ is the n-dimensional Euclidean space; the independent variable x may be any continuously and monotonely changing variable, for instance time t or one of the components of process $\mathbf{y}(x)$ that satisfies this condition. This problem was posed in [1] and is of great practical importance.

In the one-dimensional case, this problem was considered in [2], [3], and [4] for Markov diffusion processes. Namely, the exact expressions were obtained for the desired probability in a situation where the one-dimensional process $y(t) \equiv y_t$ is a solution to the stochastic differential equation

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_t,$$

where μ and $\sigma > 0$ are constants and $W(t) \equiv W_t$ is the Wiener process.

The mathematical results obtained in [2], [3], and [4] made it possible to propose an information technology within which the probability of falling a share price below the minimum acceptable level and the probability of the share price going beyond the established corridor can be calculated.

For non-Markovian smooth processes and the special case of domain Q (when Q is a half-space in $\mathbb{R}^n$), the problem posed in [1] was considered in [5] and [6]. The mathematical results obtained in [5] and [6] were used to calculate the accuracy and safety of landing an aircraft on a ship (see [7]).

Conference paper [5] and article [6] propose different approaches to estimating the desired probability. The current paper considers the one-dimensional case and concretizes the general results from [5].

II. PRELIMINARIES

We consider the following problem. Let $y(x)$ be a continuous random process and let h be a given number. We will consider process $y(x)$ defined on the semiinterval $(x_0, x'']$, $x_0 \geq -\infty$, and satisfying the condition

$$\lim_{x\to x_0} \mathbf{P}\{y(x) > h\} = 1. \tag{1}$$

Choose arbitrary $x' \in (x_0, x'')$. Define the events L and Z as follows:

$$L = \Big\{ \exists\, \hat{x} \in (x_0, x''] \ \ \forall x \in (x_0, \hat{x}) \ \ y(x) > h \Big\} \tag{2}$$

and

$$Z = \left\{ \begin{array}{r} \exists\, \tilde{x} \in (x', x'') \ \ \forall x \in (x_0, \tilde{x}) \ \ y(x) > h; \\ y(\tilde{x}) = h \end{array} \right\}.$$

Event L consists in the fact that the initial section of the trajectory $y(x)$ is located above a given level h. Event Z consists in the fact that the first achievement of a given level h by the process $y(x)$ occurs at some moment $\tilde{x}$ from a given interval (x', x''). We are to find the conditional probability $\mathbf{P}\{Z|L\}$ of event Z given that event L has occurred[1].

Following [5], [6], and [8], we denote by $G_h(x_0, x'')$ the set of scalar functions continuous on $[x_0, x'']$ or $(x_0, x'']$ (depending on the set of values for variable x that we consider) that do not identically equal h on any subinterval inside interval (x_0, x''). For functions from $G_h(x_0, x'')$, we use the concepts "crossing of level h", "touching of level h", "upcrossing of level h", and "downcrossing of level h" in accordance with their definitions given in [5], [6], and [8].

We will assume that with probability 1 sample functions $y(x)$ belong to the set $G_h(x_0, x'')$ and do not touch the level h, and the average[2] number of crossings $N(x_0, x'')$ of level h by process $y(x)$ on the interval (x_0, x'') is finite. Then, taking into account condition (1), it is easy to see that $\mathbf{P}\{Z|L\} = \mathbf{P}\{Z\}$, i.e., instead of bounds on the conditional probability $\mathbf{P}\{Z|L\}$ we can prove bounds on the unconditional probability $\mathbf{P}\{Z\}$.

By $N^+(x_1, x_2)$ denote the average number of upcrossings of level h by the process $y(x)$ on the interval (x_1, x_2), by $N^-(x_1, x_2)$ denote the average number of downcrossings of level y by the process $y(x)$ on the interval (x_1, x_2). For the process $y(x)$, we denote by $m_h(x_1, x_2)$ the average number of local maxima higher than the level h on the interval (x_1, x_2). The following result is established in [5].

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[1] We note that if x_0 is finite then all results shown below also hold if instead of processes defined on the semiinterval $(x_0, x'']$ and satisfying condition (1) we consider processes defined on the segment $[x_0, x'']$ and satisfying condition $\mathbf{P}\{y(x_0) > h\} = 1$; in this case, event L will be defined by formula $L = \{y(x_0) > h\}$ instead of formula (2).

[2] The term "average" means the mathematical expectation of the corresponding random variable.

Theorem. *Suppose that* a) *with probability* 1 *sample functions* $y(x)$ *belong to the set* $G_h(x_0, x'')$ *and do not touch the level* h *on the interval* (x_0, x''), $N(x_0, x'') < \infty$; b) $\mathbf{P}\{y(x') = h\} = 0$; c) *condition* (1) *holds. Let* x^* *be any point from the interval* (x_0, x'') *such that* $\mathbf{P}\{y(x^*)=h\}=0$, $m_h(x^*, x'')<\infty$. *Then*

$$N^-(x', x'') - \Big(N^+(x_0, x^*) + m_h(x^*, x'')\Big) \le \mathbf{P}\{Z\},$$
$$\mathbf{P}\{Z\} \le N^-(x', x''). \tag{3}$$

If the process $y(x)$ is differentiable in mean-square, then the numbers $N^+(x_0, x'')$ and $N^-(x', x'')$ can be calculated by the Rice formulas (see, e.g., [8])

$$N^+(x_0, x'') = \int_{x_0}^{x''} dx \int_0^{\infty} v f_x(h, v) dv \tag{4}$$

and

$$N^-(x', x'') = -\int_{x'}^{x''} dx \int_{-\infty}^{0} v f_x(h, v) dv. \tag{5}$$

Here $f_x(h, v) = f_x(u, v)\big|_{u=h}$, $f_x(u, v)$ is the joint distribution density of random values $y(x)$ and $y'(x)$, where $y'(x)$ is the derivative in mean-square of process $y(x)$.

The number $m_h(x^*, x'')$ can be calculated by the Rice formula (see, e.g., [8])

$$m_h(x^*, x'') = -\int_{x^*}^{x''} dx \int_h^{\infty} du \int_{-\infty}^{0} z w_x(u, 0, z) dz, \tag{6}$$

where $w_x(u, v, z)$ is the joint distribution density of random values $y(x), y'(x), y''(x)$. Here $y'(x)$ is the first derivative in mean-square of process $y(x)$ and $y''(x)$ is the second derivative in mean-square of process $y(x)$.

III. SPECIALIZATION FOR THE GAUSSIAN PROCESS

We consider the case when

$$y(x) = a_1 x + a_0 + \eta(x),\ x \in (-\infty, x''],$$

where $a_1 < 0$ and a_0 are constants, $\eta(x)$ is a stationary centered Gaussian process with continuous realizations and a normalized correlation function

$$r(\tau) \equiv \mathbf{E}\{\eta(x)\eta(x+\tau)\}/\sigma^2,$$

where σ^2 is the variance of processes $y(x)$ and $\eta(x)$, $\mathbf{E}$ is a mathematical expectation sign. If there exists a finite second derivative $r''(0)$, then the process $y(x)$ is differentiable in mean-square and we can use Rice's formulas (4) and (5) for calculating the numbers N^+ and N^-. If there exists a finite fourth derivative $r^{IV}(0)$, then the process $y(x)$ is twice differentiable in mean-square and the number m_h can be calculated by the Rice formula (6).

In the case under consideration, the conditions of Theorem hold and for every $x^* \in (-\infty, x'')$ we have

$$N^-(x', x'') - \Big(N^+(-\infty, x^*) + m_h(x^*, x'')\Big) \le \mathbf{P}\{Z\},$$
$$\mathbf{P}\{Z\} \le N^-(x', x''), \tag{3$'$}$$

where $N^+(-\infty, x^*) = \lim\limits_{x\to-\infty} N^+(x, x^*)$.

By σ_1^2 denote the variance of process $y'(x)$; by σ_2^2, the variance of process $y''(x)$; and by ρ, the correlation coefficient of random variables $y(x)$ and $y''(x)$. Note that

$$\sigma_1^2 = -\sigma^2 r''(0),\ \sigma_2^2 = \sigma^2 r^{IV}(0),\ \rho = r''(0)/\sqrt{r^{IV}(0)}.$$

It follows from these relations that the correlation coefficient ρ can only take negative values because we consider $\sigma > 0$, $\sigma_1 > 0$, and $\sigma_2 > 0$, i.e., $\sigma \ne 0$, $\sigma_1 \ne 0$, and $\sigma_2 \ne 0$.

In our case of process $y(x)$, we have

$$f_x(u, v) = \frac{1}{2\pi\sigma\sigma_1} \exp\left\{-\frac{(u-a_0-a_1x)^2}{2\sigma^2} - \frac{(v-a_1)^2}{2\sigma_1^2}\right\},$$

$$w_x(u, v, z) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(v-a_1)^2}{2\sigma_1^2}\right\} \frac{1}{2\pi\sigma\sigma_2\sqrt{1-\rho^2}} \cdot$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(u-a_0-a_1x)^2}{\sigma^2} - 2\rho\frac{(u-a_0-a_1x)z}{\sigma\sigma_2} + \frac{z^2}{\sigma_2^2}\right]\right\}.$$

Suppose for definiteness that $h=0$, $a_0=0$, and $x''>0$ (see Fig. 1). Then the Z event means that the first achievement of

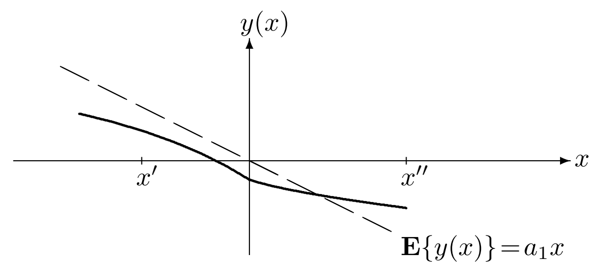

Fig. 1. The case when $h=0$ and $a_0=0$.

the zero level by process $y(x) = a_1 x + \eta(x)$ occurs at some point from the interval (x', x'').

By definition, put[3]

$$\alpha = -a_1/\sigma_1 \text{ and } \beta = -a_1 x''/\sigma.$$

Then, by formulas (4), (5), and (6), we obtain

$$N^+(-\infty, x^*) = \left[\frac{1}{\sqrt{2\pi}\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right] \Phi\left(\beta\frac{x^*}{x''}\right), \tag{4$'$}$$

$$N^-(x', x'') = \left[\frac{1}{\sqrt{2\pi}\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} + \Phi(\alpha)\right]\left[\Phi(\beta) - \Phi\left(\beta\frac{x'}{x''}\right)\right], \tag{5$'$}$$

and

$$m_h(x^*, x'') = -\frac{\beta}{2\pi\rho\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} \cdot$$
$$\int_{x^*/x''}^{1} \left[\Phi\left(-\frac{\beta}{\sqrt{1-\rho^2}}v\right) - \rho\Phi\left(-\frac{\rho\beta v}{\sqrt{1-\rho^2}}\right) \exp\left\{-\frac{\beta^2 v^2}{2}\right\}\right] dv, \tag{6$'$}$$

[3] Note that the physical meaning of parameter α is that it lets us judge how the inclination of realizations of process $y(x)$ differs from the inclination of its mathematical expectation: this difference is large for the small α and this difference is small for the large α.

where the following notation is introduced:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\{-x^2/2\} dx.$$

IV. ACCURACY OF ESTIMATES FOR THE GAUSSIAN PROCESS

Let us rewrite $(3')$ in the form

$$N^-(x', x'') - f(x^*) \le \mathbf{P}\{Z\} \le N^-(x', x''),$$

where

$$f(x^*) = N^+(-\infty, x^*) + m_h(x^*, x''),\ x^* \in (-\infty, x''].$$

Let's introduce a dimensionless variable $w = x^*/x''$. Then estimates $(3')$ take the form

$$N^-(x', x'') - f(w) \le \mathbf{P}\{Z\} \le N^-(x', x''), \qquad (3'')$$

where $w \in (-\infty, 1]$ and

$$f(w) = N^+(-\infty, w) + m_h(w, 1),$$

$$N^+(-\infty, w) = \left[\frac{1}{\sqrt{2\pi}\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right] \Phi(\beta w),$$

$$m_h(w, 1) = -\frac{\beta}{2\pi\rho\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\}$$

$$\cdot \int_w^1 \left[\Phi\left(-\frac{\beta}{\sqrt{1-\rho^2}} v\right) - \rho\Phi\left(-\frac{\rho\beta v}{\sqrt{1-\rho^2}}\right) \exp\left\{-\frac{\beta^2 v^2}{2}\right\}\right] dv.$$

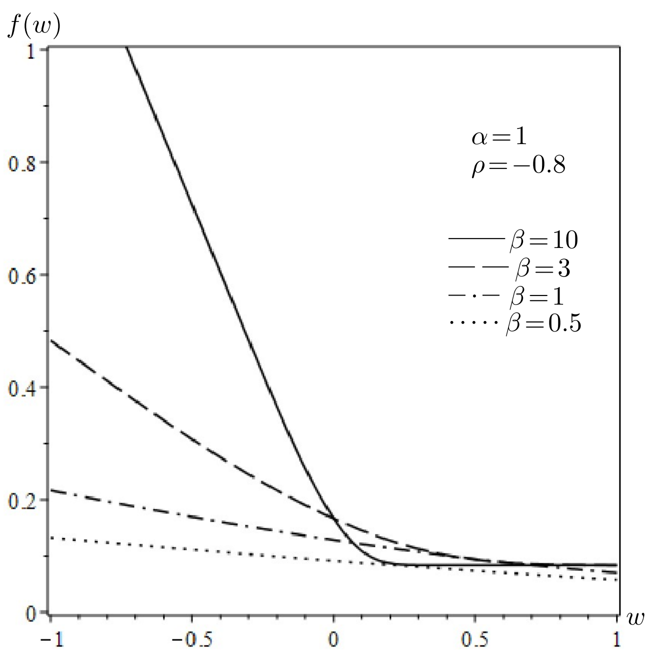

Fig. 2. Curves $f(w)$ for different β when $\alpha=1$ and $\rho=-0.8$.

Let us study the function $f(w)$ on the interval $(-\infty, 1]$: the smaller the values of $f(w)$, the more accurate the estimate for the probability $\mathbf{P}\{Z\}$ will be given by inequalities $(3'')$.

We will study the behavior of the function $f(w)$ depending on three parameters: α, β, and ρ. We will be interested in the interval $w \in [-1, 1]$ because when $w < -1$, based on the physical meaning of the numbers $N^+(-\infty, w)$ and $m_h(w, 1)$, it is obvious that the function $f(w)$ will be monotonically decreasing[4], so that the minimum values of the function $f(w)$ that interest us will be achieved when $w \in [-1, 1]$.

Fig. 2 shows the behavior of function $f(w)$ for different β when $\alpha = 1$ and $\rho = -0.8$. For any combination of values α, β, and ρ, it turned out that the function $f(w)$ is monotonically decreasing on the interval $(-\infty, 1]$. This conclusion follows from the expression for the derivative $f'(w)$. Using the well-known formula for the derivative of an integral with variable limits

$$\frac{d}{dt} \int_{\phi_1(t)}^{\phi_2(t)} g(v) dv = -g(\phi_1(t)) \frac{d\phi_1(t)}{dt} + g(\phi_2(t)) \frac{d\phi_2(t)}{dt},$$

we obtain the following expression for the derivative $f'(w)$ of the function $f(w)$:

$$\frac{df(w)}{dw} = \left[\frac{1}{\sqrt{2\pi}\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right] \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\beta^2 w^2}{2}\right\} \beta$$
$$+ \frac{\beta}{2\pi\rho\alpha} \exp\left\{-\frac{\alpha^2}{2}\right\} \left[\Phi\left(-\frac{\beta w}{\sqrt{1-\rho^2}}\right)\right.$$
$$\left. - \rho\Phi\left(-\frac{\rho\beta w}{\sqrt{1-\rho^2}}\right) \exp\left\{-\frac{\beta^2 w^2}{2}\right\}\right].$$

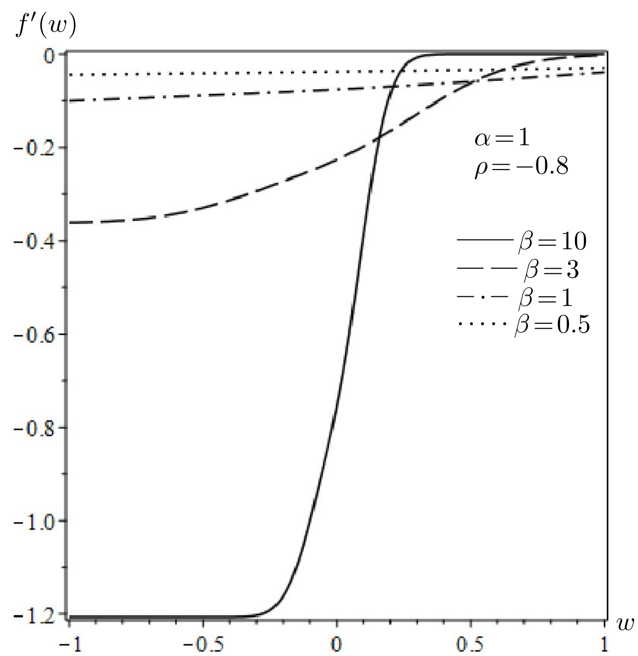

Fig. 3. Curves $f'(w)$ for different β when $\alpha=1$ and $\rho=-0.8$.

Fig. 3 shows the behavior of derivative $f'(w)$ for different β when $\alpha=1$ and $\rho=-0.8$. Numerical calculations showed that $f'(w) < 0$ when $w \in (-\infty, 1]$. Therefore, the minimum values of function $f(w)$ are achieved at $w = 1$. But the nature of the change in the derivative $f'(w)$ can vary greatly depending on parameters α, β, and ρ. In particular, in the

[4]This fact is confirmed by numerical calculations.

case $\beta=10$, $f'(w)\approx 0$ (see fig. 3) and $f(w)$ (see fig. 2) practically does not change when the number w changes in the interval $[0.5, 1]$[5].

V. CONCLUSION

In this paper, for the Gaussian process $y(x)$, we performed an analysis of accuracy of estimates $(3'')$ for the probability $\mathbf{P}\{Z\}$; namely, in the case $\alpha=1$ and $\rho=-0.8$, we investigated the accuracy of these estimates depending on the choice of point x^*, as well as the parameter β characterizing the considered Gaussian process $y(x)$. The accuracy of the estimates is determined by the value $f(w), w\in(-\infty,1]$ (see $(3'')$), where the number w is uniquely determined by point x^*. The smaller $f(w)$, the more accurately the probability $\mathbf{P}\{Z\}$ is determined. It turned out that, *for the considered Gaussian process* $y(x)$, the minimum value of $f(w)$ is obtained with $w=1$ for any β.

In the future, we plan to continue studying the accuracy of estimates $(3'')$ for the considered Gaussian process $y(x)$, varying the parameters α and ρ. We also plan to study the accuracy of estimates $(3'')$ *for non-Gaussian processes* $y(x)$.

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[5]Preliminary calculations showed that the conclusions drawn for pair $(\alpha=1,\ \rho=-0.8)$ also hold true for other pairs (α,ρ). Also, it turned out that a) for a fixed α value, the results change little with changes in the ρ parameter, and b) for a fixed ρ value, the accuracy of estimates is improving (i.e., the $f(w)$ value is decreasing) with an increase in the α parameter. The author plans to explore this topic in more detail in future papers.