# A new cutting plane algorithm for the min $s$-$t$ cut problem

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***Abstract*—We describe a new LP model for min $s$-$t$ cut that is the smallest known LP formulation with respect to the number of variables ($O(n)$ in our model vs $O(n^2)$ or even more for models in the literature). On the other hand, our model has $O(n!)$ constraints, but we describe a fast cutting-plane algorithm to effectively separate over the exponentially many constraints. Overall, we prove that our model can be solved in polynomial time and, in fact, preliminary computational experiments suggest that the number of iterations needed for convergence is sublinear. Given that $O(n)$ constraints are added in each iteration, the final LP appears to be of the smallest known size, with a linear number of variables and a close to linear number of constraints.**

***Index Terms*—Minimum cut problem, Linear programming, Cutting planes**

## I. INTRODUCTION

The minimum $s$-$t$ cut problem is one of the classical combinatorial optimization problems and it has been widely studied in the literature [1], [5]. In this problem, we are given an undirected graph $G = (V, E)$, two specific nodes $s, t \in V$ and non-negative costs $c_{ij}$ for all edges $ij \in E$. In this paper we will assume $V = \{1, \ldots, n\}$, $s = 1$ and $t = n$. The problem consists in partitioning $V$ into two subsets, $S \ni s$ and $T \ni t$, so that the costs of the edges between $S$ and $T$ are as small as possible. This objective can be stated as the minimization of $c(S,T) := \sum\{c_{ij} : ij \in E, (i \in S \wedge j \in T) \vee (j \in S \wedge i \in T)\}$, where $S \cap T = \emptyset$, $S \cup T = V$.

The minimum $s$-$t$ cut problem is closely related to the *maximum $s$-$t$ flow* problem [1]. In this case, $G$ is viewed as a flow network in which the values $c_{ij}$ describe the capacities of the edges, and the maximum amount of some commodity must be shipped along the network from $s$ to $t$. A well-known theorem states that the maximum $s$-$t$ flow and the minimum $s$-$t$ cut values are the same. Furthermore, given the solution to the optimal maximum $s$-$t$ flow problem, it is easy to compute the minimum $s$-$t$ cut so that, from an algorithmic point of view, the two problems are substantially equivalent. There are many quite effective combinatorial algorithms for these problems, such as [3], [4], [8]. In addition to combinatorial algorithms, the minimum $s$-$t$ cut problem can also be solved through linear programming by solving the following model:

$$
\begin{array}{rcl}
ORG: & \min & \sum_{ij \in E} c_{ij} x_{ij} \\
x_{ij} & \geq & h_i - h_j \quad \forall ij \in E \\
x_{ij} & \geq & h_j - h_i \quad \forall ij \in E \\
h_1 & = & 0 \\
h_n & = & 1 \\
h_i & \in & [0,1] \quad \forall i \in V \\
x_{ij} & \in & \mathbb{R}^+ \quad \forall ij \in E
\end{array}
$$

In the model $ORG$, standing for "original", there are node variables $h_i \in [0, 1]$ and edge variables $x_{ij}$. It turns out that each optimal solution, say $(\boldsymbol{h}^*, \boldsymbol{x}^*)$, which is an extreme point of the feasible region of $ORG$ is naturally integer. For such a solution, $\boldsymbol{h}^*$ identifies the sets $S_{\boldsymbol{h}^*} := \{i : h_i^* = 0\}$ and $T_{\boldsymbol{h}^*} := \{i : h_i^* = 1\}$ corresponding to a minimum $s$-$t$ cut. Where $m = |E|$, the model has $\Theta(m+n)$ variables and $\Theta(m)$ constraints.

### A. Main contribution and work in progress

In this paper, we will introduce a new linear programming formulation for the min $s$-$t$ cut problem. Our formulation has $O(n)$ variables and $O(n!)$ constraints, and we solve it via a cutting-plane algorithm, i.e., an approach used in combinatorial optimization for exponential-size linear programs. The procedure keeps a working LP containing only a (small) subset of constraints, and then determines if its optimal solution satisfies also the missing constraints. If that's not the case, one or more missing constraints (called *cutting planes*) are then added to the model for a new iteration. Preliminary computational experiments (not reported here) suggest that the total number of cutting planes added to obtain the optimal solution is a small multiple of $n$. We do not have any proof on the worst-case scenario for this approach, but, from empirical evidence, we believe that a conservative bound of $O(n^2)$ cutting planes could be proved. The studying of the theoretical worst-case and average-case performance of our cutting-plane algorithm is the subject of our ongoing research.

### B. Notation

The edges of the graph will be denoted by $ij \in E$, where, by convention, $i < j$. We also define $E_n := \{ij : 1 \le i < j \le n\}$ as the set of all ordered pairs of nodes. Given a subset $Q \subseteq V$, by $\delta(Q)$ we denote the set of edges with one endpoint in $Q$ and one out of $Q$. Also, $\delta(i)$ is defined to be $\delta(\{i\})$ for any node $i \in V$. If $F \subseteq E$, we define $c(F) := \sum_{ij \in F} c_{ij}$. For any $a \in \mathbb{R}$, the function $\text{sgn}(a)$ is defined as

$$\text{sgn}(a) = \begin{cases} +1 & \text{if } a \ge 0 \\ -1 & \text{if } a < 0 \end{cases}$$

Notice that $\text{sgn}(a)a = |a|$ for each $a \in \mathbb{R}$. Since we will be often considering vectors $\boldsymbol{h} = (h_1, \dots, h_n)$ such that $h_1 = 0$ and $h_n = 1$, for convenience, we define the following specific subsets of $\{0,1\}^n$ and $[0,1]^n$

$${}_0\{0,1\}_1^n := \{0\} \times \{0,1\}^{n-2} \times \{1\}$$

and

$${}_0[0,1]_1^n := \{0\} \times [0,1]^{n-2} \times \{1\}.$$

### C. Paper organization

The remainder of this paper is organized as follows. In Section II, we describe the new linear programming formulation for minimum $s$-$t$ cut, as well as the basic cutting-plane approach for its solution. In Section III, we introduce a correspondence between inequalities and permutations, showing that all the inequalities of the model are facet-defining, and that this list of facet-defining inequalities is complete. Section IV is devoted to some conclusions and directions for further research. Being this a "work in progress" submission, for space reasons most proofs of our claims have been omitted and they are postponed to a journal version of the work.

## II. A CUTTING-PLANE MODEL

The minimum $s$-$t$ cut problem can be modeled by the following, non-linear, mathematical programming problem:

$$\begin{aligned} MP: \quad & \min \quad z \\ h_1 &= 0 \\ h_n &= 1 \\ z &= \sum_{ij \in E} c_{ij} |h_i - h_j| \quad \forall ij \in E \qquad (1) \\ h_i &\in \{0,1\} \quad \forall i \in V \end{aligned}$$

In this section, we describe how to turn $MP$ into an equivalent integer linear programming model, $ILP$, and then prove that its linear programming relaxation, $LPR$, is naturally integer.

As already mentioned, each vector $\bar{\boldsymbol{h}} \in {}_0\{0,1\}_1^n$ identifies an $s$-$t$ cut, namely $S_{\bar{\boldsymbol{h}}} = \{i : \bar{h}_i = 0\}$ and $T_{\bar{\boldsymbol{h}}} = \{i : \bar{h}_i = 1\}$. The value of this cut can be written as

$$c(S_{\bar{\boldsymbol{h}}}, T_{\bar{\boldsymbol{h}}}) = \sum_{ij \in E} c_{ij} |\bar{h}_j - \bar{h}_i| = \sum_{ij \in E} c_{ij} \, \text{sgn}(\bar{h}_j - \bar{h}_i)(\bar{h}_j - \bar{h}_i).$$

Notice that, for each $i, j \in V$,

$$\text{sgn}(\bar{h}_j - \bar{h}_i)(\bar{h}_j - \bar{h}_i) \ge \beta \, (\bar{h}_j - \bar{h}_i) \quad \text{for } \beta \in \{-1, +1\}. \quad (2)$$

Let us call a vector $\boldsymbol{b} \in \{-1,+1\}^{E_n}$ a *signing*. Given a signing $\boldsymbol{b}$, we also extend its scope to pairs $uv$ with $u > v$ by defining $b_{uv} := -b_{vu}$, making $\boldsymbol{b}$ anti-symmetric. By the previous argument and the non-negativity of $c_{ij}$, we get that for every $\bar{\boldsymbol{h}} \in {}_0\{0,1\}_1^n$ and every signing $\boldsymbol{b}$, the value

$$LB(\boldsymbol{b}, \bar{\boldsymbol{h}}) := \sum_{ij \in E} c_{ij} \, b_{ij} (\bar{h}_j - \bar{h}_i)$$

is a lower bound to the value of the cut induced by $\bar{\boldsymbol{h}}$, which means

$$c(S_{\bar{\boldsymbol{h}}}, T_{\bar{\boldsymbol{h}}}) \ge LB(\boldsymbol{b}, \bar{\boldsymbol{h}}) \qquad \forall \bar{\boldsymbol{h}} \in {}_0\{0,1\}_1^n, \forall \boldsymbol{b} \in \{-1,+1\}^{E_n}.$$

It is important to notice that for all $\bar{\boldsymbol{h}} \in {}_0\{0,1\}_1^n$ (and hence, for all possible $s$-$t$ cuts) there exists at least one signing $\bar{\boldsymbol{b}}$ such that

$$c(S_{\bar{\boldsymbol{h}}}, T_{\bar{\boldsymbol{h}}}) = LB(\boldsymbol{b}, \bar{\boldsymbol{h}}). \quad (3)$$

This signing is unique if all components of $\bar{\boldsymbol{h}}$ are distinct values. In particular, $\boldsymbol{b}^{\bar{\boldsymbol{h}}}$ defined by

$$b_{ij}^{\bar{\boldsymbol{h}}} := \text{sgn}(\bar{h}_j - \bar{h}_i) \qquad \forall 1 \le i < j \le n \quad (4)$$

achieves (3).

We can now define the following integer linear program, which is the relaxation of $MP$ obtained by replacing constraint (1) with constraints (6) below:

$$\begin{aligned} ILP: \quad & \min \quad z && (5) \\ h_1 &= 0 \\ h_n &= 1 \\ z &\ge LB(\boldsymbol{b}, \boldsymbol{h}) \quad \forall \boldsymbol{b} \in \{-1,+1\}^{E_n} && (6) \\ h_i &\in \{0,1\} \quad \forall i \in V \end{aligned}$$

In this model the real variable $z$ represents the value of a minimum $s$-$t$ cut, while the binary variables $h_i$ identify the cut.

*Claim 1:* Let $(\boldsymbol{h}^*, z^*)$ be an optimal solution of $ILP$. Then $(S_{\boldsymbol{h}^*}, T_{\boldsymbol{h}^*})$ is a minimum $s$-$t$ cut, and $z^*$ is its value.

Let $LPR$ be the linear programming relaxation of $ILP$. We want to prove that solving $LPR$ is equivalent to solving $ILP$.

Given any $\boldsymbol{h} \in {}_0[0,1]_1^n$, we can define a vector $\boldsymbol{x}^{\boldsymbol{h}} \in [0,1]^{E_n}$ by setting, for each $ij \in E$,

$$x_{ij}^{\boldsymbol{h}} := |h_j - h_i| = \text{sgn}(h_j - h_i)(h_j - h_i). \quad (7)$$

This way, a solution $(\boldsymbol{h}, \boldsymbol{x}^{\boldsymbol{h}})$ of $ORG$ can be obtained. Furthermore, we can define $z^{\boldsymbol{h}}$ by setting

$$z^{\boldsymbol{h}} := \max_{\boldsymbol{b} \in \{-1,+1\}^{E_n}} LB(\boldsymbol{b}, \boldsymbol{h}) = \sum_{ij \in E} c_{ij} \, \text{sgn}(h_j - h_i)(h_j - h_i). \quad (8)$$

This way, a solution $(\boldsymbol{h}, z^{\boldsymbol{h}})$ of $LPR$ can be obtained. Notice that the $\boldsymbol{b}$ yielding $z^{\boldsymbol{h}}$ is in fact $\boldsymbol{b}^{\boldsymbol{h}}$ defined in (4). We make the following important observation:

*Observation 1:* For any $\boldsymbol{h} \in {}_0[0,1]_1^n$, we have $z^{\boldsymbol{h}} = \boldsymbol{c} \cdot \boldsymbol{x}^{\boldsymbol{h}}$.

*Claim 2:* Let $(\boldsymbol{h}^*, z^*)$ be an extreme point for the feasible set of $LPR$. Then $z^* = z^{\boldsymbol{h}^*}$.

*Theorem 2.1:* Let $(\boldsymbol{h}^*, z^{\boldsymbol{h}^*})$ be an extreme point for the feasible set of $LPR$. Then $(\boldsymbol{h}^*, x^{\boldsymbol{h}^*})$ is an extreme point for the feasible set of $ORG$. In particular, this implies $\boldsymbol{h}^* \in {}_0\{0,1\}_1^n$.

**Proof:** (Sketch) Since for each fixed signing $\boldsymbol{b}$, $LB(\boldsymbol{b}, \cdot)$ is linear function of $\boldsymbol{h}$, we can show that for each $\lambda, \mu \in \mathbb{R}^+$ and each $\boldsymbol{h}', \boldsymbol{h}'' \in {}_0[0,1]_1^n$,

$$\max_{\boldsymbol{b}} LB(\boldsymbol{b}, \lambda\boldsymbol{h}' + \mu\boldsymbol{h}'') \leq \lambda \max_{\boldsymbol{b}} LB(\boldsymbol{b}, \boldsymbol{h}') + \mu \max_{\boldsymbol{b}} LB(\boldsymbol{b}, \boldsymbol{h}'')$$

and, in turn,

$$z^{\lambda\boldsymbol{h}' + \mu\boldsymbol{h}''} \leq \lambda z^{\boldsymbol{h}'} + \mu z^{\boldsymbol{h}''}. \tag{9}$$

Now, the proof proceeds by contradiction. Suppose $(\boldsymbol{h}^*, \boldsymbol{x}^{\boldsymbol{h}^*})$ is not an extreme point for $ORG$, but it is a convex combination of two distinct solutions, i.e., $(\boldsymbol{h}^*, \boldsymbol{x}^{\boldsymbol{h}^*}) = \alpha(\boldsymbol{h}', \boldsymbol{x}') + (1-\alpha)(\boldsymbol{h}'', \boldsymbol{x}'')$. Componentwise, for each $ij \in E$,

$$x_{ij}^{\boldsymbol{h}^*} := |h_j^* - h_i^*| = \alpha x'_{ij} + (1-\alpha) x''_{ij}. \tag{10}$$

Furthermore,

$$|h_j^* - h_i^*| = \left|\alpha(h'_j - h'_i) + (1-\alpha)(h''_j - h''_i)\right| \tag{11}$$

Knowing that $x'_{ij} \geq |h'_j - h'_i|$ and $x''_{ij} \geq |h''_j - h''_i|$ , for each $ij \in E$, by a chain of inequalities and by using (10) we can show that in fact

$$x'_{ij} = |h'_j - h'_i|, \qquad x''_{ij} = |h''_j - h''_i| \qquad \forall ij \in E, \tag{12}$$

which means $\boldsymbol{x}' = \boldsymbol{x}^{\boldsymbol{h}'}$ and $\boldsymbol{x}'' = \boldsymbol{x}^{\boldsymbol{h}''}$. This, in turn, implies $\boldsymbol{h}' \neq \boldsymbol{h}''$.

From (9) and by using Observation 1, we can show that

$$z^{\boldsymbol{h}^*} = z^{\alpha\boldsymbol{h}' + (1-\alpha)\boldsymbol{h}''} \leq \alpha z^{\boldsymbol{h}'} + (1-\alpha) z^{\boldsymbol{h}''} = z^{\boldsymbol{h}^*}$$

so that equality holds throughout. We thus obtain

$$(\boldsymbol{h}^*, z^{\boldsymbol{h}*}) = \alpha(\boldsymbol{h}', z^{\boldsymbol{h}'}) + (1-\alpha)(\boldsymbol{h}'', z^{\boldsymbol{h}''})$$

with all these vectors being solutions of $LPR$. Since $\boldsymbol{h}' \neq \boldsymbol{h}''$, $(\boldsymbol{h}^*, z^{\boldsymbol{h}^*})$ would not be an extreme point for $LPR$, a contradiction.

Finally, $\boldsymbol{h}$ is naturally integer at all extreme points $(\boldsymbol{h}, \boldsymbol{x})$ of $ORG$ [6]. □

*Corollary 2.2:* Let $(\boldsymbol{h}^*, z^{\boldsymbol{h}^*})$ be an optimal solution of $LPR$, which is an extreme point. Then $(S_{\boldsymbol{h}^*}, T_{\boldsymbol{h}^*})$ is a minimum $s$-$t$ cut, and $z^{\boldsymbol{h}^*}$ is its value.

### A. *From general $\{-1,+1\}$-signings to permutations*

An anti-symmetric signing $\boldsymbol{b}$ is *transitive* if

$$(b_{ij} = +1) \wedge (b_{jk} = +1) \implies b_{ik} = +1 \ \forall \text{distinct } i,j,k \in V.$$

*Claim 3:* For every $\boldsymbol{h} \in {}_0[0,1]_1^n$ the signing $\boldsymbol{b}^{\boldsymbol{h}}$ defined by (4) is transitive. Furthermore, $\max_{\boldsymbol{b}} LB(\boldsymbol{b}, \boldsymbol{h}) = LB(\boldsymbol{b}^{\boldsymbol{h}}, \boldsymbol{h})$.

*Lemma 2.3:* The model $LPR$, in which inequalities (6) are restricted to transitive signings only, is equivalent to $LPR$.

Notice that by the definition of $\boldsymbol{b}^{\boldsymbol{h}}$, for every $\boldsymbol{h} \in {}_0[0,1]_1^n$ and each $1 < k < n$, we have $b_{1k}^{\boldsymbol{h}} = b_{kn}^{\boldsymbol{h}} = +1$. Let $\mathcal{B}^T$ be the set of all transitive signings such that $b_{1k} = b_{kn} = +1$ for each $1 < k < n$. In view of Corollary 2.3, we come to the final definition of our new model for minimum $s$-$t$ cut, which is the following:

$$\begin{array}{rrcll}
NEW: & & \min & z & \\
& h_1 & = & 0 & \\
& h_n & = & 1 & \\
& z & \geq & \displaystyle\sum_{ij \in E} c_{ij}\, b_{ij}(h_j - h_i) & \forall \boldsymbol{b} \in \mathcal{B}^T \quad (13) \\
& h_i & \in & [0,1] & \forall i \in V
\end{array}$$

In constraints (13) of $NEW$, we have written out the formulation of $LB(\boldsymbol{b}, \boldsymbol{h})$ for better readability. By computing the individual coefficients $C_k^{\boldsymbol{b}}$ for each $h_k$ variable, each constraint (13) can be written as

$$z \geq \sum_{k \in V} C_k^{\boldsymbol{b}} h_k \tag{14}$$

where

$$\begin{aligned}
C_k^{\boldsymbol{b}} &:= \sum_{\substack{i<k \\ ik \in E}} c_{ik} b_{ik} - \sum_{\substack{j>k \\ kj \in E}} c_{kj} b_{kj} \\
&= \sum_{\substack{i<k \\ ik \in E \\ b_{ik}=+1}} c_{ik} - \sum_{\substack{i<k \\ ik \in E \\ b_{ik}=-1}} c_{ik} + \sum_{\substack{j>k \\ kj \in E \\ b_{kj}=-1}} c_{kj} - \sum_{\substack{j>k \\ kj \in E \\ b_{kj}=+1}} c_{kj}
\end{aligned}$$

We can notice that the coefficients $C_k^{\boldsymbol{b}}$ are related both to the costs of the edges and to the degrees of the nodes. In particular, they can become very large, in absolute value, for dense graphs or graphs in which the edge costs vary in a wide range.

## III. Transitive signings, permutations and facets

In order to count the number of inequalities of model $NEW$, we show that there exists a one-to-one correspondence between transitive signings and permutations $\pi$ of $\{1, \ldots, n\}$, with $\pi(1) = 1$, $\pi(n) = n$. Let $S_{1n}^n$ be the set of such permutations.

*Lemma 3.1:* There is a one-to-one correspondence between transitive signings of $\mathcal{B}^T$ and the permutations of $S_{1n}^n$.

*Corollary 3.2:* The number of transitive signings is $(n-2)!$.

Since model $NEW$ has $(n-2)!$ inequalities (13), we want to solve it by a cutting-plane algorithm. At the generic iteration, only a subset of inequalities (13) will be present in the current model, corresponding to a subset $\mathcal{B}'$ of transitive signings. Let $(\boldsymbol{h}^*, z^*)$ be the optimal solution for the current LP. We then compute the strongest transitive signing for the current $\boldsymbol{h}^*$, namely, $\boldsymbol{b}^{\boldsymbol{h}^*}$, and check if $z^* \geq LB(\boldsymbol{b}^{\boldsymbol{h}^*}, \boldsymbol{h}^*)$. If this is the case, since Claim 3 gives $LB(\boldsymbol{b}^{\boldsymbol{h}^*}, \boldsymbol{h}^*) \geq LB(\boldsymbol{b}, \boldsymbol{h}^*)$ for all transitive signings $\boldsymbol{b}$, there are no inequalities violated by $(\boldsymbol{h}^*, z^*)$, which is then optimal for $NEW$. Otherwise, we have found a violated inequality, and we add the constraint corresponding to $\boldsymbol{b}^{\boldsymbol{h}^*}$ to the model. This separation algorithm has cost $O(m)$, since we need to compute $b_{ij}^{\boldsymbol{h}^*}$ only for $ij \in E$.

*Theorem 3.3:* The exponential-size model $NEW$ can be solved in polynomial time.

To strengthen our cutting-plane approach, we also made the following consideration. The cut added to the model is $z \geq LB(\boldsymbol{b}^{\boldsymbol{h}^*}, \boldsymbol{h})$. For some $\boldsymbol{h}$, the right-hand side of this inequality might be less than zero which, clearly, is a useless lower bound to $z$. In order to make sure that the inequalities we add guarantee that $z \geq 0$ at each iteration, we define and utilize the concept of *reverse* signings.

Given a transitive signing $\boldsymbol{b}$, its reverse signing $\overleftarrow{\boldsymbol{b}}$ is defined as follows, for each $1 \leq i < j \leq n$:

$$\overleftarrow{b}_{ij} = \begin{cases} b_{1j} & \text{if } i = 1 \\ -b_{ij} & \text{if } 1 < i < j < n \\ b_{in} & \text{if } j = n \end{cases}$$

Notice that, for any $\boldsymbol{h} \in {}_0[0,1]_1^n$,

$$LB(\boldsymbol{b}, \boldsymbol{h}) < 0 \implies LB(\overleftarrow{\boldsymbol{b}}, \boldsymbol{h}) > 0$$

and hence at least one of the two inequalities $z \geq LB(\boldsymbol{b}, \boldsymbol{h}) \forall \boldsymbol{h}$ and $z \geq LB(\overleftarrow{\boldsymbol{b}}, \boldsymbol{h}) \forall \boldsymbol{h}$ would be non-trivial. Therefore, in our cutting-plane phase, we compute both $\boldsymbol{b}^{\boldsymbol{h}^*}$ and $\overleftarrow{\boldsymbol{b}}^{\boldsymbol{h}^*}$ and add, if any, the inequalities corresponding to both signings.

It is not difficult to see that if $\pi = (\pi(1), \pi(2), \cdots, \pi(n-1), \pi(n))$ is the permutation, as described in Lemma 3.1, corresponding to a transitive signing $\boldsymbol{b}$, then $\overleftarrow{\boldsymbol{b}}$ must also be transitive and the permutation corresponding to $\overleftarrow{\boldsymbol{b}}$ is $\pi' := (\pi(1), \pi(n-1), \cdots, \pi(2), \pi(n))$.

We close the section by stating the second main theorem relative to our model, i.e., that each inequality (13) in $NEW$ is facet defining.

*Theorem 3.4:* Let $P$ be the polyhedron corresponding to the feasible solutions of $NEW$, and let $\hat{\boldsymbol{b}} \in \mathcal{B}^T$ be a transitive signing. Then the inequality

$$z \geq \sum_{ij \in E} c_{ij}\, \hat{b}_{ij}(h_j - h_i) \tag{15}$$

defines a facet of $P$.

## IV. Conclusions

In this paper we have introduced a new LP formulation for the minimum $s$-$t$ cut problem. Our formulation is the smallest known in terms of variables ($O(n)$) but has an exponential number of constraints. However, we have shown how to quickly separate over the exponential set of constraints. In our preliminary experiments we have observed that, most of the times, only a small number of rounds are required to converge to the optimal solution and, overall, our approach turns out to be very effective on many instances. Studying the number of iterations that are required, on average and/or in the worst case, is at the moment the focus of our research.

Besides the implementation of the basic model, we are also investigating a variant which may represent a considerable strenthening. In this variant, we have introduced $n$ additional real-valued variables $z_j$, one for each $j = 1, \ldots, n$. Informally, each $z_j$ represents the contribution to the value of the cut given by the edges incident to $j$, that is, $z_j = c\,(\delta(j) \cap \delta(S))$, where $(S, T)$ is an $s$-$t$ cut. If $\boldsymbol{h} \in {}_0\{0,1\}_1^n$ represents the cut, then the values of $z_j$ correspond to

$$z_j = \sum_{\substack{i=1 \\ ij \in E}}^{j-1} c_{ij}|h_j - h_i| + \sum_{\substack{i=j+1 \\ ji \in E}}^{n} c_{ji}|h_i - h_j|. \tag{16}$$

We have run some preliminary experiments, both for the model $NEW$ and for its variant, on some special and general instances. The simplest graphs consisted of either a random Hamilton path from $s$ to $t$, or $\sqrt{n}$ disjoint paths of roughly length $\sqrt{n}$ from $s$ to $t$, where every node (other than $s$ and $t$) is contained in a path. These simple problems where each edge has cost 1, have trivial solutions, but are some of the more interesting problems for our models.

More complex problems were created by starting with one of these simple graphs, but replacing edge costs of 1 with integer edge costs relatively high and randomly assigned, e.g., between 900 and 1000; then the weights of some number of random edges (not just those with positive weights at this point) are increased by a random integer between 5 and 15. Finally, to avoid having the optimal minimum cut separate a single node from the rest, the weights of edges adjacent to $s$ or $t$ are increased by $1000n$.

These complex problems were studied in three types: those with a linear number ($10n$) of edges randomly increased, those with $n^{1.5}$ edges randomly increased, and those with every edge randomly increased.

From our preliminary experiments, we notice a large difference in the number of separation steps required by $NEW$ and by the variant with the $z_j$ variables, that can result much faster.

The number of separation steps appears to be constant, or near constant (certainly sub-linear), in the average case, but we have noticed that outliers can require more steps, and we have yet to find a way of proving an upper bound on the number of steps required, or whether such a bound even exists.

The most difficult problems for the algorithm are problems where the underlying graph is simply an $s$-$t$ path, as these graphs represent facet-defining inequalities in the original minimum $s$-$t$ cut problem [7]. Of these problems, $s$-$t$ Hamilton paths required the most effort.

In the future, we intend to run a much larger set of extensive experiments, with instances of various sizes generated according to standard random models or taken from the repositories in the literature. The goal will be that of comparing our versions of the models to each other and to the models from the literature, both with respect to their running times and to the size of the resulting linear programs.

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