# Hyper-Exponential Dynamic Regressor Extension and Mixing with Covariance Adaptation

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***Abstract*—This paper develops a diagonal covariance Dynamic Regressor Extension and Mixing (DREM) estimator for linearly parameterized regressions. The diagonal gain structure preserves the scalar decoupling property of DREM, while an excitation-driven amplifier reshapes the adaptation clock generated by the mixed regressor. In the nominal case, an explicit solution reveals a sharp threshold, separating residual convergence, standard DREM-type convergence, and hyper-exponential decay with respect to accumulated mixed-regressor excitation. A positive diagonal covariance lower bound prevents gain collapse and recovers convergence under the standard DREM condition. A sustained mixed-regressor excitation yields hyper-exponential decay in physical time. Convergence under interval excitation and robustness to bounded measurement noise are also analyzed, making explicit the trade-off between fast adaptation and noise amplification. A simulation example illustrates the relative effectiveness and the robustness of the technique.**

## I. Introduction

Parameter estimation for linearly parameterized systems is a central tool in adaptive control, identification, and observer design. Classical gradient and least-squares estimators provide asymptotic or exponential convergence under persistent excitation (PE) [1], [2]. In many practical settings, however, excitation is intermittent, transient, or generated only indirectly by closed-loop operation. This has motivated the development of online estimators that retain the recursive simplicity of gradient algorithms while relaxing the PE requirement.

Dynamic regressor extension and mixing (DREM) provides an effective framework for this purpose [3]. By filtering the original regression and applying an adjugate-based mixing operation, DREM transforms a vector regression into scalar parameter regressions driven by a common mixed regressor $\Delta(t)$. This scalarization yields componentwise error dynamics and replaces PE of the original regressor by weaker excitation conditions on $\Delta$. The convergence properties of DREM have been further clarified in continuous and discrete time [4]–[6], and several recent variants have addressed least-squares extensions, excitation preservation, robustness, and accelerated convergence [7]–[13].

Despite these developments, standard DREM still evolves according to the accumulated mixed-regressor energy $\int_0^t \Delta^2(\tau)d\tau$. Thus, while DREM relaxes the type of excitation needed for convergence, the adaptation clock itself remains directly tied to this accumulated energy. The objective of this paper is to reshape this clock through a diagonal covariance-gain mechanism and an excitation-dependent amplifier. The diagonal restriction is essential: it preserves the scalar decoupling property of DREM, whereas a full matrix gain would generally recouple the scalar channels.

The proposed estimator introduces an auxiliary state $r$ satisfying $\dot{r} = \alpha\Delta^2$ and multiplies the diagonal DREM adaptation gains by $e^{r(t)}$. In the no-floor case, the resulting scalar error equations admit an explicit solution and reveal three regimes separated by the threshold $\alpha = 2\gamma$: residual convergence, standard DREM-type convergence, and hyper-exponential decay with respect to accumulated mixed-regressor excitation. A diagonal covariance-floor modification prevents gain collapse and restores convergence under the standard DREM condition $\Delta \notin L_2$. Under sustained mixed-regressor excitation, the convergence becomes hyper-exponential in physical time.

The main contribution is therefore an excitation-driven, diagonal covariance-floor DREM estimator that accelerates convergence while preserving the componentwise scalar structure of DREM. The paper also establishes convergence under interval excitation and gives a robustness analysis for bounded output noise, making explicit the trade-off between fast adaptation and noise amplification. A single vector DREM example with matched initial adaptation coefficients is used to compare the proposed method with standard DREM and covariance-DREM and to illustrate sensitivity to measurement noise and the covariance floor.

The paper is organized as follows. Section II formulates the regression problem and recalls the DREM construction. Section III introduces the linear covariance-gain update. Section IV develops the diagonal hyper-exponential DREM estimator, its covariance-floor modification, and the interval-excitation result. Section V studies robustness to bounded measurement noise. Section VI presents the comparison and numerical validation, and Section VII concludes the paper.

## II. Problem Formulation and DREM Construction

Consider the linearly parameterized scalar regression

$$y(t) = \phi^\top(t)\theta, \tag{1}$$

where $y(t) \in \mathbb{R}$ is measured, $\phi(t) \in \mathbb{R}^p$ is known, bounded, and piecewise continuous, and $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector. The estimation error is denoted by

$$\tilde{\theta} := \theta - \hat{\theta}.$$

Throughout the nominal development, the regression (1) is assumed to be noise-free. The effect of bounded measurement disturbances is considered later.

**Problem 1.** Given the regression (1), design a causal online estimator for $\theta$, using only $y(t)$, $\phi(t)$, and filtered versions of these signals. The objective is to construct an estimator whose convergence condition is expressed in terms of the scalar DREM mixed regressor, rather than the vector regressor $\phi$, and whose convergence rate can subsequently be shaped by covariance and excitation-dependent gain mechanisms.

The dynamic regressor extension and mixing procedure begins by processing (1) through $p-1$ causal exponentially stable linear filters. The first-order filters

$$\mathcal{H}_i(s) = \frac{\lambda_i}{s+\lambda_i}, \qquad \lambda_i > 0, \qquad i = 1, \dots, p-1,$$

are used. Equivalently, the filtered signals are generated by

$$\dot{y}_{f_i} = -\lambda_i y_{f_i} + \lambda_i y, \qquad \dot{\phi}_{f_i} = -\lambda_i \phi_{f_i} + \lambda_i \phi,$$

with zero initial filter states. Since $\theta$ is constant and the filters are linear, one has

$$y_{f_i}(t) = \phi_{f_i}^\top(t)\theta, \qquad i = 1, \dots, p-1.$$

If nonzero filter initial conditions are used, the same identities hold up to exponentially decaying transients; these transients are omitted in the nominal construction.

Stacking the original and filtered regressions gives the extended regression

$$Y_e(t) = \Phi_e(t)\theta, \tag{2}$$

where

$$Y_e := \operatorname{col}\big(y, y_{f_1}, \dots, y_{f_{p-1}}\big), \qquad \Phi_e := \begin{bmatrix} \phi^\top \\ \phi_{f_1}^\top \\ \vdots \\ \phi_{f_{p-1}}^\top \end{bmatrix}.$$

The DREM mixed regressor and mixed output are defined by

$$\Delta(t) := \det\big(\Phi_e(t)\big), \qquad \mathcal{Y}(t) := \operatorname{adj}\big(\Phi_e(t)\big)Y_e(t).$$

Here $\operatorname{adj}(\cdot)$ denotes the adjugate matrix. Using

$$\operatorname{adj}(\Phi_e)\Phi_e = \det(\Phi_e)I_p,$$

one obtains

$$\mathcal{Y}(t) = \Delta(t)\theta. \tag{3}$$

Thus the vector regression (2) is transformed into the family of scalar regressions

$$\mathcal{Y}_i(t) = \Delta(t)\theta_i, \qquad i = 1, \dots, p. \tag{4}$$

No pointwise nonsingularity of $\Phi_e(t)$ is required for the identity (3). The determinant $\Delta(t)$ may vanish at isolated times or over intervals. The relevant excitation condition is instead an integral condition on the scalar mixed regressor. Specifically,

$$\Delta \notin L_2 \iff \int_0^\infty \Delta^2(t)\,dt = \infty.$$

This is the basic DREM replacement for persistent excitation of the original vector regressor.

For reference, the standard DREM gradient estimator associated with (4) is

$$\dot{\hat{\theta}}_i = \gamma_i \Delta(t)\big(\mathcal{Y}_i(t) - \Delta(t)\hat{\theta}_i(t)\big), \qquad \gamma_i > 0.$$

Since

$$\mathcal{Y}_i - \Delta\hat{\theta}_i = \Delta(\theta_i - \hat{\theta}_i) = \Delta\tilde{\theta}_i,$$

the scalar error dynamics are

$$\dot{\tilde{\theta}}_i = -\gamma_i \Delta^2(t)\tilde{\theta}_i. \tag{5}$$

Consequently,

$$\tilde{\theta}_i(t) = \tilde{\theta}_i(0)\exp\left(-\gamma_i \int_0^t \Delta^2(\tau)\,d\tau\right).$$

Therefore, each component converges to zero whenever $\Delta \notin L_2$ [3], [6].

Equation (5) is the baseline scalar error model. The remainder of the paper modifies the effective adaptation coefficient multiplying $\Delta^2(t)$ by introducing covariance and excitation-dependent gain dynamics. The goal is to preserve the scalar DREM structure while accelerating the convergence clock generated by the mixed regressor.

## III. Linear Covariance-Gain Update

Before introducing the hyper-exponential DREM estimator, we recall a linear matrix gain update for the base regression (1). Let $\hat{\theta} \in \mathbb{R}^p$ be the estimate, $\tilde{\theta} := \theta - \hat{\theta}$, and

$$e := y - \phi^\top\hat{\theta} = \phi^\top\tilde{\theta}.$$

Let $M(t) = M^\top(t) > 0$ denote a matrix adaptation gain. Consider

$$\dot{\hat{\theta}} = \beta M\phi e, \tag{6}$$

$$\dot{M} = -\gamma\big(M\phi\phi^\top + \phi\phi^\top M\big), \tag{7}$$

where $\beta, \gamma > 0$. Then

$$\dot{\tilde{\theta}} = -\beta M\phi\phi^\top\tilde{\theta}. \tag{8}$$

The update (7) is linear in $M$. It is used here as a gain-shaping mechanism rather than as the standard continuous-time RLS law. This distinction is useful because, after DREM mixing, the same structure reduces to scalar gain dynamics on the decoupled DREM channels.

**Proposition 1.** *Assume $M(0) = M^\top(0) > 0$. Let $\Psi(t)$ solve*

$$\dot{\Psi} = -\gamma\phi(t)\phi^\top(t)\Psi, \qquad \Psi(0) = I.$$

*Then*

$$M(t) = \Psi(t)M(0)\Psi^\top(t), \tag{9}$$

*so $M(t) = M^\top(t) > 0$ for all $t \geq 0$. Moreover,*

$$\lambda_{\min}(M(t)) \geq \lambda_{\min}(M(0))\exp\left(-2\gamma\int_0^t \|\phi(\tau)\|^2 d\tau\right). \tag{10}$$

## IV. Hyper-Exponential DREM

We now apply the covariance-gain idea to the scalar DREM regressions (4). The key design restriction in this section is that the gain matrix is kept diagonal. This is not merely a simplification: it is what preserves the componentwise scalar structure of DREM. Thus, let

$$M_d(t) := \operatorname{diag}\big(\rho_1(t), \dots, \rho_p(t)\big), \qquad \rho_i(0) > 0.$$

Define the DREM prediction error

$$e_D := \mathcal{Y} - \Delta\hat{\theta}, \qquad e_{D,i} = \mathcal{Y}_i - \Delta\hat{\theta}_i = \Delta\tilde{\theta}_i,$$

where $\tilde{\theta}_i := \theta_i - \hat{\theta}_i$. The diagonal hyper-DREM estimator is

$$\begin{aligned} \dot{\hat{\theta}}_i &= \beta e^{r(t)}\rho_i(t)\Delta(t)e_{D,i}(t), \\ \dot{\rho}_i &= -2\gamma\Delta^2(t)\rho_i(t), \qquad\qquad i = 1, \dots, p, \\ \dot{r} &= \alpha\Delta^2(t), \end{aligned} \tag{11}$$

where $\alpha, \beta, \gamma > 0$. Since $e_{D,i} = \Delta\tilde{\theta}_i$, each error channel satisfies

$$\dot{\tilde{\theta}}_i = -\beta e^{r(t)}\rho_i(t)\Delta^2(t)\tilde{\theta}_i. \tag{12}$$

No cross-coupling is introduced: the common scalar regressor $\Delta(t)$ and the common amplifier $r(t)$ affect all channels, but the adaptation gains $\rho_i(t)$ remain scalar.

Define the accumulated mixed-regressor excitation

$$I(t) := \int_0^t \Delta^2(\tau)\, d\tau. \tag{13}$$

Then

$$r(t) = r(0) + \alpha I(t), \qquad \rho_i(t) = \rho_i(0)e^{-2\gamma I(t)}.$$

Hence the effective coefficient in (12) is

$$e^{r(t)}\rho_i(t) = e^{r(0)}\rho_i(0)e^{(\alpha-2\gamma)I(t)}.$$

This identity exposes the threshold $\alpha = 2\gamma$: excitation amplification dominates covariance-gain decay precisely when $\alpha > 2\gamma$.

**Theorem 1.** *Consider* (11) *and define $I(t)$ by* (13). *Let*

$$a := \alpha - 2\gamma, \qquad \kappa_i := \beta e^{r(0)}\rho_i(0),$$

*and define*

$$G_a(s) := \begin{cases} \dfrac{e^{as}-1}{a}, & a \neq 0, \\ s, & a = 0. \end{cases}$$

*Then, for every $i = 1, \dots, p$,*

$$\tilde{\theta}_i(t) = \tilde{\theta}_i(0)\exp\big(-\kappa_i G_a(I(t))\big). \tag{14}$$

*Consequently, the following regimes hold.*

1) *If $\alpha < 2\gamma$, then $G_a(I(t))$ is bounded. In particular, if $I(t) \to \infty$, then*
$$\lim_{t\to\infty} \tilde{\theta}_i(t) = \tilde{\theta}_i(0)\exp\left(-\frac{\kappa_i}{2\gamma - \alpha}\right).$$
*Thus the no-floor diagonal estimator generally converges only to a residual value unless $\tilde{\theta}_i(0) = 0$.*
2) *If $\alpha = 2\gamma$, then*
$$\tilde{\theta}_i(t) = \tilde{\theta}_i(0)e^{-\kappa_i I(t)}.$$
*Hence $\tilde{\theta}_i(t) \to 0$ whenever $\Delta \notin L_2$, equivalently $I(t) \to \infty$. This is the standard DREM convergence clock with effective gain $\kappa_i$.*
3) *If $\alpha > 2\gamma$, then*
$$|\tilde{\theta}_i(t)| = |\tilde{\theta}_i(0)| \exp\left[-\frac{\kappa_i}{\alpha - 2\gamma}\left(e^{(\alpha-2\gamma)I(t)} - 1\right)\right].$$
*Therefore $\tilde{\theta}_i(t) \to 0$ whenever $\Delta \notin L_2$, and the decay is hyper-exponential with respect to the integrated excitation $I(t)$. If, in addition, there exist $T \geq 0$ and $\mu > 0$ such that $\Delta^2(t) \geq \mu$ for all $t \geq T$, then the decay is hyper-exponential in physical time.*

*Proof.* Using the explicit solutions for $r$ and $\rho_i$, the error equation (12) becomes

$$\dot{\tilde{\theta}}_i = -\kappa_i e^{aI(t)}\Delta^2(t)\tilde{\theta}_i.$$

Since $\dot{I} = \Delta^2$, the function $G_a$ satisfies

$$\frac{d}{dt}G_a(I(t)) = e^{aI(t)}\Delta^2(t).$$

Therefore the scalar linear equation integrates directly to (14). The three cases follow by inspecting the limit of $G_a(s)$ as $s \to \infty$. When $a > 0$ and $\Delta^2(t) \geq \mu > 0$ for $t \geq T$, one has $I(t) \geq I(T) + \mu(t - T)$, which gives a bound of the form

$$|\tilde{\theta}_i(t)| \leq C_i|\tilde{\theta}_i(0)| \exp\Big(-c_i e^{(\alpha-2\gamma)\mu t}\Big)$$

for suitable constants $C_i, c_i > 0$. This proves the stated hyper-exponential decay in physical time. □

### *A. Hyper-Exponential DREM with Diagonal Covariance Floor*

The diagonal estimator in (11) exposes the acceleration mechanism, but it also reveals a limitation. If the scalar gain $\rho_i(t)$ decays too aggressively, the effective adaptation clock may saturate. To prevent this loss of adaptation, we introduce a strictly diagonal covariance-gain floor.

Let

$$M_d(t) = \operatorname{diag}(\rho_1(t), \dots, \rho_p(t)),\ P_\infty = \operatorname{diag}(\rho_{\infty,1}, \dots, \rho_{\infty,p}),$$

where $\rho_i(0) \geq \rho_{\infty,i} > 0, i = 1, \dots, p$. The diagonal covariance-floor hyper-DREM estimator is

$$\begin{aligned} \dot{\hat{\theta}}_i &= \beta e^{r(t)}\rho_i(t)\Delta(t)e_{D,i}(t), \\ \dot{\rho}_i &= -2\gamma e^{r(t)}\Delta^2(t)\big(\rho_i(t) - \rho_{\infty,i}\big), \qquad i = 1, \dots, p, \\ \dot{r} &= \alpha\Delta^2(t), \end{aligned} \tag{15}$$

where $e_{D,i} = \mathcal{Y}_i - \Delta\hat{\theta}_i = \Delta\tilde{\theta}_i$ and $\alpha, \beta, \gamma > 0$. Equivalently,

$$\dot{\hat{\theta}} = \beta e^r M_d \Delta e_D, \qquad \dot{M}_d = -2\gamma e^r \Delta^2(M_d - P_\infty),$$

with both $M_d$ and $P_\infty$ diagonal for all time.

Define the amplified excitation clock

$$S(t) := \int_0^t e^{r(\tau)}\Delta^2(\tau)\, d\tau. \tag{16}$$

Since $\dot{r} = \alpha\Delta^2$, one has

$$S(t) = \frac{e^{r(t)} - e^{r(0)}}{\alpha}. \tag{17}$$

**Lemma 1.** *For every $i = 1, \dots, p$, the solution of* (15) *satisfies*

$$\rho_i(t) = \rho_{\infty,i} + \big(\rho_i(0) - \rho_{\infty,i}\big)e^{-2\gamma S(t)}. \tag{18}$$

*Consequently, $\rho_{\infty,i} \le \rho_i(t) \le \rho_i(0)$, $t \ge 0$.*

*Proof.* The $i$th floor dynamics can be written as

$$\dot{\rho}_i = -2\gamma \dot{S}(t)\big(\rho_i - \rho_{\infty,i}\big).$$

Integration gives (18). Since $\rho_i(0) \ge \rho_{\infty,i} > 0$, the stated bounds follow directly. $\square$

**Definition 1.** The origin of a time-varying system is called hyper-exponentially stable if there exist constants $c_1, c_2, c_3 > 0$ such that

$$\|x(t)\| \le c_1\|x(0)\| \exp\big(-c_2(e^{c_3 t} - 1)\big), \qquad t \ge 0.$$

**Theorem 2.** *Consider the diagonal covariance-floor estimator* (15). *Assume*

$$\int_0^\infty \Delta^2(t)\, dt = \infty.$$

*Then $\tilde{\theta}(t) \to 0$. More precisely, for each $i = 1, \dots, p$,*

$$\tilde{\theta}_i(t) = \tilde{\theta}_i(0) \exp\Bigg(-\beta\rho_{\infty,i}S(t) - \frac{\beta\big(\rho_i(0) - \rho_{\infty,i}\big)}{2\gamma} \times \big(1 - e^{-2\gamma S(t)}\big)\Bigg). \tag{19}$$

*In particular, with $\rho_\infty := \min_{1 \le i \le p} \rho_{\infty,i}$, the vector error satisfies*

$$\|\tilde{\theta}(t)\| \le \|\tilde{\theta}(0)\| \exp\left(-\frac{\beta\rho_\infty}{\alpha}\big(e^{r(t)} - e^{r(0)}\big)\right). \tag{20}$$

*If, in addition, there exist $T \ge 0$ and $\mu > 0$ such that $\Delta^2(t) \ge \mu$ for all $t \ge T$, then the origin is hyper-exponentially stable.*

*Proof.* Using $e_{D,i} = \Delta\tilde{\theta}_i$, the error dynamics are

$$\dot{\tilde{\theta}}_i = -\beta e^r \rho_i \Delta^2 \tilde{\theta}_i = -\beta \dot{S}(t)\rho_i(t)\tilde{\theta}_i.$$

Substituting (18) gives

$$\dot{\tilde{\theta}}_i = -\beta\dot{S}(t)\left[\rho_{\infty,i} + \big(\rho_i(0) - \rho_{\infty,i}\big)e^{-2\gamma S(t)}\right]\tilde{\theta}_i.$$

Integrating with respect to $S$ yields (19). Since

$$\rho_i(t) \ge \rho_{\infty,i} \ge \rho_\infty,$$

one also obtains

$$|\tilde{\theta}_i(t)| \le |\tilde{\theta}_i(0)| e^{-\beta\rho_\infty S(t)}.$$

Using (17) gives (20).

The condition $\int_0^\infty \Delta^2(t)dt = \infty$ implies $r(t) = r(0) + \alpha\int_0^t \Delta^2(\tau)d\tau \to \infty$. Hence $S(t) \to \infty$, and (20) gives $\tilde{\theta}(t) \to 0$.

Finally, if $\Delta^2(t) \ge \mu > 0$ for all $t \ge T$, then

$$r(t) \ge r(T) + \alpha\mu(t - T), \qquad t \ge T.$$

Substitution into (20) gives constants $c_1, c_2, c_3 > 0$ such that

$$\|\tilde{\theta}(t)\| \le c_1\|\tilde{\theta}(0)\| \exp\big(-c_2(e^{c_3 t} - 1)\big), \qquad t \ge 0.$$

Thus the origin is hyper-exponentially stable. $\square$

### *B. Convergence Under Interval Excitation*

The condition $\Delta \notin L_2$ is weaker than persistent excitation of the original vector regressor $\phi$. It is useful, however, to record a simple sufficient condition stated directly in terms of the scalar DREM mixed regressor. The result below shows that the diagonal covariance-floor estimator converges under recurring finite-window excitation of $\Delta$.

**Theorem 3.** *Consider the diagonal covariance-floor estimator* (15). *Assume that there exist constants $T > 0$, $c > 0$, and an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$, with $t_k \to \infty$, such that*

$$\int_{t_k}^{t_k+T} \Delta^2(\tau)\, d\tau \ge c, \qquad k \in \mathbb{N}. \tag{21}$$

*Then*

$$\int_0^\infty \Delta^2(t)\, dt = \infty, \qquad r(t) \to \infty, \qquad S(t) \to \infty,$$

*where $S(t)$ is defined in* (16). *Consequently, $\tilde{\theta}(t) \to 0$. More precisely, with $\rho_\infty := \min_i \rho_{\infty,i}$,*

$$\|\tilde{\theta}(t)\| \le \|\tilde{\theta}(0)\| \exp\big(-\beta\rho_\infty S(t)\big). \tag{22}$$

*Proof.* The only point requiring care is that the intervals in (21) need not be disjoint. Since $t_k \to \infty$, one can extract a subsequence $\{t_{k_j}\}_{j\in\mathbb{N}}$ such that

$$t_{k_{j+1}} \ge t_{k_j} + T, \qquad j \in \mathbb{N}.$$

Hence the intervals $[t_{k_j}, t_{k_j} + T]$ are pairwise disjoint. For any $N \in \mathbb{N}$ and any $t \ge t_{k_N} + T$,

$$\int_0^t \Delta^2(\tau)\, d\tau \ge \sum_{j=1}^N \int_{t_{k_j}}^{t_{k_j}+T} \Delta^2(\tau)\, d\tau \ge Nc.$$

Since $N$ is arbitrary, $\int_0^\infty \Delta^2(t)dt = \infty$. From $\dot{r} = \alpha\Delta^2$, it follows that

$$r(t) = r(0) + \alpha\int_0^t \Delta^2(\tau)d\tau \to \infty.$$

Using (17),

$$S(t) = \frac{e^{r(t)} - e^{r(0)}}{\alpha} \to \infty.$$

The error bound (22) follows directly from Theorem 2, and therefore $\tilde{\theta}(t) \to 0$. $\square$

## V. Robustness with Respect to Bounded Noise

We now examine the effect of bounded measurement noise. Suppose that the measured output is

$$y(t) = \phi^\top(t)\theta + \nu(t), \qquad |\nu(t)| \le \bar{\nu}. \tag{23}$$

The regressor $\phi$ is assumed to be measured exactly. Hence the extended regressor $\Phi_e$, the determinant $\Delta$, and the adjugate $\operatorname{adj}(\Phi_e)$ are unchanged by the output noise. Only the extended output is perturbed. Let

$$d_e := \operatorname{col}\left(\nu, \nu_{f_1}, \dots, \nu_{f_{p-1}}\right), \qquad \nu_{f_i} := \mathcal{H}_i[\nu].$$

Then the extended regression becomes

$$Y_e = \Phi_e\theta + d_e.$$

After DREM mixing,

$$\mathcal{Y} = \operatorname{adj}(\Phi_e)Y_e = \Delta\theta + \eta, \qquad \eta := \operatorname{adj}(\Phi_e)d_e. \tag{24}$$

Since the filters are BIBO stable and $\Phi_e$ is bounded whenever $\phi$ is bounded, the mixed disturbance is bounded. Thus there exist constants $\bar{\eta}_i \ge 0$ such that

$$|\eta_i(t)| \le \bar{\eta}_i, \qquad i = 1, \dots, p.$$

For zero filter initial conditions and first-order filters with unit $L_\infty$-gain, one may take $\|\eta(t)\| \le C_{\mathrm{adj}}\sqrt{p}\,\bar{\nu}$, where $C_{\mathrm{adj}} := \sup_{t\ge 0} \|\operatorname{adj}(\Phi_e(t))\|$. Nonzero filter initial conditions only add exponentially decaying terms and can be absorbed into $\eta$.

Consider the diagonal covariance-floor estimator (15). Since

$$e_{D,i} = \mathcal{Y}_i - \Delta\hat{\theta}_i = \Delta\tilde{\theta}_i + \eta_i,$$

the noisy error dynamics are

$$\dot{\tilde{\theta}}_i = -\beta e^r \rho_i \Delta^2 \tilde{\theta}_i - \beta e^r \rho_i \Delta \eta_i, \qquad i = 1, \dots, p. \tag{25}$$

Thus the same gain that accelerates convergence also multiplies the noise-injection term. This is the fundamental speed–robustness trade-off of the proposed design.

Define $a_i(t) := \beta e^{r(t)}\rho_i(t)\Delta^2(t)$ and, for $t \ge s$,

$$\Gamma_i(t, s) := \exp\left(-\int_s^t a_i(\tau)\, d\tau\right). \tag{26}$$

**Proposition 2.** *For every solution of* (25)*,*

$$\begin{aligned} |\tilde{\theta}_i(t)| \le& \Gamma_i(t,0)|\tilde{\theta}_i(0)| \\ &+ \beta\bar{\eta}_i \int_0^t \Gamma_i(t,\tau)e^{r(\tau)}\rho_i(\tau)|\Delta(\tau)|\, d\tau. \end{aligned} \tag{27}$$

*If, in addition, $\Delta \notin L_2$, then the homogeneous factor $\Gamma_i(t,0)$ converges to zero.*

*Proof.* Equation (25) is scalar and linear. Its variation-of-constants formula gives

$$\tilde{\theta}_i(t) = \Gamma_i(t,0)\tilde{\theta}_i(0) - \beta \int_0^t \Gamma_i(t,\tau)e^{r(\tau)}\rho_i(\tau)\Delta(\tau)\eta_i(\tau)\, d\tau,$$

which implies (27). For the covariance-floor estimator, Lemma 1 gives $\rho_i(t) \ge \rho_{\infty,i} > 0$. Therefore

$$\int_0^t a_i(\tau)d\tau \ge \beta\rho_{\infty,i} \int_0^t e^{r(\tau)}\Delta^2(\tau)d\tau = \beta\rho_{\infty,i}S(t),$$

where $S(t)$ is defined in (16). If $\Delta \notin L_2$, then $r(t) \to \infty$ and $S(t) \to \infty$ by (17). Hence $\Gamma_i(t,0) \to 0$. □

A uniform ultimate bound requires an additional condition preventing the mixed regressor from becoming arbitrarily small relative to the mixed disturbance.

**Proposition 3.** *Assume the hypotheses of Proposition 2. Suppose there exist $T \ge 0$ and $\underline{\Delta} > 0$ such that*

$$|\Delta(t)| \ge \underline{\Delta}, \qquad t \ge T.$$

*Then, for every $t \ge T$,*

$$|\tilde{\theta}_i(t)| \le \Gamma_i(t,T)|\tilde{\theta}_i(T)| + \frac{\bar{\eta}_i}{\underline{\Delta}}\big(1 - \Gamma_i(t,T)\big). \tag{28}$$

*Consequently,*

$$\limsup_{t\to\infty} |\tilde{\theta}_i(t)| \le \frac{\bar{\eta}_i}{\underline{\Delta}}. \tag{29}$$

*Moreover, with*

$$\bar{\eta} := \left(\sum_{i=1}^p \bar{\eta}_i^2\right)^{1/2},$$

*the vector error satisfies*

$$\limsup_{t\to\infty} \|\tilde{\theta}(t)\| \le \frac{\bar{\eta}}{\underline{\Delta}}. \tag{30}$$

*If, in addition, $\Delta^2(t) \ge \mu > 0$ for all $t \ge T$, then the approach to the residual ball is hyper-exponential.*

*Proof.* For $t \ge T$, apply the variation-of-constants formula from $T$ to $t$:

$$\begin{aligned} |\tilde{\theta}_i(t)| \le& \Gamma_i(t,T)|\tilde{\theta}_i(T)| \\ &+ \beta\bar{\eta}_i \int_T^t \Gamma_i(t,\tau)e^{r(\tau)}\rho_i(\tau)|\Delta(\tau)|\, d\tau. \end{aligned}$$

Since $|\Delta(\tau)| \ge \underline{\Delta}$ for $\tau \ge T$, $\beta e^{r(\tau)}\rho_i(\tau)|\Delta(\tau)|\bar{\eta}_i \le \frac{\bar{\eta}_i}{\underline{\Delta}}a_i(\tau)$. Thus

$$|\tilde{\theta}_i(t)| \le \Gamma_i(t,T)|\tilde{\theta}_i(T)| + \frac{\bar{\eta}_i}{\underline{\Delta}} \int_T^t \Gamma_i(t,\tau)a_i(\tau)d\tau.$$

For fixed $t$, $\frac{\partial}{\partial\tau}\Gamma_i(t,\tau) = a_i(\tau)\Gamma_i(t,\tau)$. Therefore

$$\int_T^t \Gamma_i(t,\tau)a_i(\tau)d\tau = \Gamma_i(t,t) - \Gamma_i(t,T) = 1 - \Gamma_i(t,T),$$

which proves (28). Since $\rho_i(t) \ge \rho_{\infty,i} > 0$ and $|\Delta(t)| \ge \underline{\Delta}$ for $t \ge T$,

$$\int_T^\infty a_i(\tau)d\tau = \infty,$$

and hence $\Gamma_i(t,T) \to 0$. This proves (29). The vector bound (30) follows by summing the scalar limsup bounds.

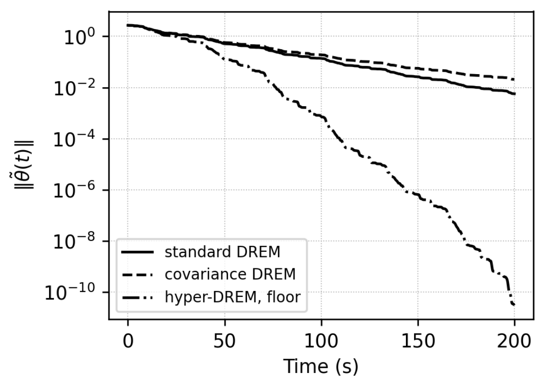


Fig. 1. Comparison of three DREM methods for the three parameter example. The proposed diagonal covariance-floor hyper-DREM law accelerates convergence through the amplified clock $S(t)$.

If $\Delta^2(t) \geq \mu > 0$ for $t \geq T$, then $r(t) \geq r(T) + \alpha\mu(t - T)$. Using $\rho_i(t) \geq \rho_{\infty,i}$,

$$\int_T^t a_i(\tau)d\tau \geq \quad \frac{\beta\rho_{\infty,i}}{\alpha}\left(e^{r(t)} - e^{r(T)}\right),$$

and the right-hand side grows at least exponentially in $t$. Hence $\Gamma_i(t, T)$ decays hyper-exponentially, so (28) gives hyper-exponential convergence to the residual ball. $\square$

## VI. Comparison and Numerical Validation

This section presents the numerical validation using the single vector example with $p = 3$:

$$\begin{aligned}\phi(t) = \mathrm{col}\big(&\sin(0.7t) + 0.35\cos(1.9t),\\ &\cos(1.1t) + 0.25\sin(2.3t),\\ &\sin(1.6t + 0.4) + 0.30\cos(2.8t)\big),\end{aligned}$$

with $\theta = \mathrm{col}(1, -2, 1.5)$ and $y(t) = \phi^\top(t)\theta + \bar{\nu}\sin(10t)$. The DREM extension uses two filters with poles $\lambda_1 = 0.5$ and $\lambda_2 = 2.5$. The compared estimators are standard DREM, diagonal covariance-DREM without floor, and the proposed diagonal covariance-floor hyper-DREM. The gains are $\beta_c = \beta_h = 5$, $\rho_0 = 1$, $\gamma_c = \gamma_h = 0.2$, $\alpha_h = 4$, $\rho_\infty = 0.08$. For fairness, the standard DREM gain is selected as $\gamma_s = \beta_h\rho_0 = 5$, so all methods have the same initial adaptation coefficient. All initial estimates are zero. Figure 1 shows the nominal comparison with $\bar{\nu} = 0$. As expected, the diagonal covariance-floor hyper-DREM estimator produces the predicted faster decay.

To address noise and tuning sensitivity, the same example is repeated with bounded output noise and different covariance floors. Define the terminal average error $E_{\mathrm{tail}} := \frac{1}{10}\int_{T_f-10}^{T_f} \|\tilde{\theta}(t)\|dt$, $T_f = 200$. Figure 2 reports $E_{\mathrm{tail}}$ for $\bar{\nu} \in \{0, 0.005, 0.01, 0.02, 0.05\}$ with $\rho_\infty = 0.08$, and for $\rho_\infty \in \{0.02, 0.05, 0.08, 0.15, 0.30\}$ with $\bar{\nu} = 0.02$. The trend is consistent with the bound in Section V.

## VII. Conclusion

A diagonal covariance-floor hyper-DREM estimator was developed for linearly parameterized regressions. The diagonal gain structure is central: it preserves the scalar decoupling property of DREM while enabling excitation-dependent amplification of the adaptation clock. Without a covariance floor, the error dynamics exhibit three regimes separated by the threshold $\alpha = 2\gamma$. With a positive diagonal floor, convergence is recovered under $\Delta \notin L_2$, and sustained excitation yields hyper-exponential decay. The interval-excitation and robustness analyses clarify the practical scope of the method, including the trade-off between faster adaptation and noise amplification.

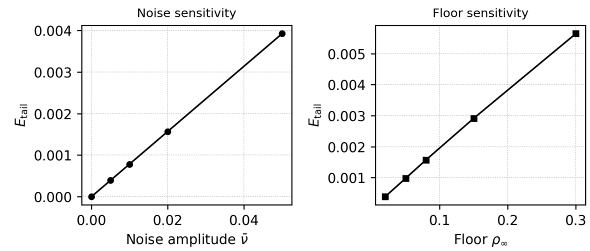


Fig. 2. Sensitivity of the proposed diagonal covariance-floor hyper-DREM. Left: terminal average error versus measurement-noise amplitude with $\rho_\infty = 0.08$. Right: terminal average error versus covariance floor with $\bar{\nu} = 0.02$.

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