Some results on scaled consensus over directed graphs

Elena Panteley Antonio Loría Alessio Iovine

Abstract— **We study the problem of controlled scaled consensus, which consists on making a network of cooperative systems achieve a multiconsensus state determined both by their initial conditions and preassigned weights. The problem is well-documented in the literature, but our main contributions reside in providing strict Lyapunov functions to establish global exponential stability of the synchronization manifold, for first and second-order systems interconnected over strongly connected graphs. We also provide some useful observations pertaining to graph theory, particularly, to Laplacians corresponding to graphs with self loops. The problem setting, as we broach it here is inspired from problems appearing in control of electrical networks.**

I. Introduction

In the realm of multi-agent systems, that is, groups of systems interconnected in a manner that each so-called agent, receives information from a set of "neighbors", consensus is the steady state in which all the systems states achieve a common equilibrium point. Therefore, consensus is, in a way, a state of synchronization. The achievement of consensus is particularly desired in the case of cooperative systems, but it is also a phenomenon that may appear spontaneously in self-organized networks of physical systems.

Achieving consensus, generally speaking, depends on the network's topology (*e.g.,* whether it is time-varying or static), the nature of the interconnections (which may be directed or undirected), and the intensity of the latter. Indeed, in the latter case, the network couplings may have different gains. In the simplest case, it is well-known that consensus is reached if and only if the network contains a spanning tree (there exists at least one node from which any node can be reached). Moreover, for simple integrator systems, the consensus equilibrium is uniquely defined by the initial conditions of the node's states. However, in other cases, such as in lack of spanning tree [1], the network may achieve a steady state of *multiconsensus* [2], [3]. In this case, subnetworks are formed, each of which achieves its own consensus equilibrium.

Closely related to multiconsensus is the so-called *scaled* consensus problem [4]. Scaled consensus is a state in which each individual node achieves an assigned proportion of a common value that depends on a weighted average of the initial conditions. Such assigned proportions correspond, roughly, to weights that factor the individual nodes *states*, and they are independent of the individual *coupling* strengths between pairs of nodes [5]. For instance, in a network of cooperative systems the weights may be assimilated to the importance of an agent's state value, while the coupling strength may be assimilated to the power of the transmission. In a more concrete setting, such as that of formation control of ground autonomous vehicles, the weights may be designed as functions of the physical distances between pairs of robots [6].

Scaled consensus also has a strong motivation that stems from electrical networks [7], but it also leads to interesting, more abstract, problems of network analysis. For instance, while the individual weights in the network have a clear interpretation in power networks, their introduction in the consensus algorithm leads to a modification of the so-called Laplacian matrix which describes the network's topology in a manner that, without actually changing the interconnections among the nodes, self-loops are added. Mathematically, this changes many properties that one often relies on in the analysis of networked systems [8], but also, the self-loops can be assimilated to grounding effects in terms of electrical networks [7].

Beyond [4], scaled consensus has been studied, *e.g.,* in [9], [10], [11]. In [9] the context is that of networks with switching topology, in a scenario in which the topology commutes from one not necessarily connected configuration to another, but they form, altogether, a connected graph—*cf.* [12]. In [10] scaled consensus is established for first- and second-order systems, over static topologies, under the condition that there exists a spanning tree; for second-order systems, the focus is on full state ("position" and "velocity") consensus. In [11] scaled consensus is addressed for strongly connected networks of first-order systems, under the assumption that the state measurements are saturated.

As in [11], [10], [9], [4] in this paper we focus on driftless systems; both of first and second order as in [11], [10], [9]. We assume that the topology is static and the networks is strongly connected, but our main contribution is that we establish scaled consensus via Lyapunov's direct method. That is, we provide for both first- and second-order systems, strict Lyapunov functions (positive definite with negative definite derivatives in the space of the synchronization errors) from which one can conclude global exponential stability of the synchronization manifold. This is significant because our results may serve as starting point to further extensions to consider the analysis of scaled consensus in networks of complex (nonlinear) systems, such as networks of oscillators [5].

The authors are with the Laboratoire des signaux et systèmes, CNRS, 3, Rue Joliot Curie, Gif sur Yvette, France. E-mail: elena.panteley@centralesupelec.fr, antonio.loria@cnrs.fr, alessio.iovine@centralesupelec.fr.
This work has been partially funded by the Agence Nationale de la Recherche (ANR) via grant ESTHER number ANR-22-CE05-0016.

II. Problem formulation and mathematical setting

Let us consider a network of interconnected systems, defined as

$$\dot{x}_i = u_i, \quad i \in \{1, 2, \ldots, N\} \tag{1a}$$

$$u_i = -\sum_{j \neq i} a_{ij}(x_i - x_j), \quad a_{ij} \geq 0 \tag{1b}$$

where, for simplicity, $x_i \in \mathbb{R}$. We assume that, in general, the interconnection coefficients $a_{ij} \neq a_{ji}$, so the underlying graph is directed and, moreover, it contains a spanning tee. That is, for (1) we have—see *e.g.,* [13],

$$\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t) = x_c, \tag{2}$$

where $x_c := v_\ell^\top x(0)$, $x := [x_1 \cdots x_N]^\top$, and v_ℓ is the left eigenvector of the Laplacian matrix $L := [\ell_{ij}] \in \mathbb{R}^{N \times N}$, associated to the unique zero eigenvalue of L, $\lambda_1(L) = 0$. The entries of L are defined as

$$\ell_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} a_{ik} & i = j \\ -a_{ij} & i \neq j, \end{cases} \tag{3}$$

where $\mathcal{N}_i \subset \mathbb{Z}$ is the set of indexes corresponding to nodes that are neighbors of the ith agent. Then, we consider the scaled consensus problem; a network is said to achieve scaled consensus if, asymptotically, the ratios between the state variables reach specified constants.

Definition 1 (Scaled consensus, [4]): We say that the states $x := [x_1 \cdots x_N]^\top$ of a network achieve scaled consensus (at x_m) with weights $w = [w_1 \cdots w_N]^\top$, where $w_i > 0$ for all $i \in \{1, 2, \ldots, N\}$, if

$$\lim_{t \to \infty} x_i(t) = w_i x_m, \quad \forall\, i \in \{1, 2, \ldots, N\}. \tag{4}$$

•

Remark 1: The scaled-consensus problem may be recast as that of the asymptotic stabilization of the manifold

$$S := \left\{ x \in \mathbb{R}^N : \frac{x_1}{w_1} = \frac{x_2}{w_2} = \cdots = \frac{x_N}{w_N} \right\}. \tag{5}$$

To see this, consider an arbitrarily fixed point $x_m \in \mathbb{R}$. Then, (4) holds if

$$\begin{bmatrix} x_1^* \\ \vdots \\ x_N^* \end{bmatrix} = \begin{bmatrix} w_1 x_m \\ \vdots \\ w_N x_m \end{bmatrix} \tag{6}$$

is an attractive equilibrium for the network system. On the other hand, (6) is equivalent to

$$\frac{x_1^*}{w_1} = \frac{x_2^*}{w_2} = \cdots = \frac{x_N^*}{w_N} = x_m. \tag{7}$$

That is, the asymptotic stability of the manifold S in (5) implies the achievement of scaled consensus. The interest of this observation is that the problem may be addressed via Lyapunov stability theory. •

Problem statement: In the present setting, we are interested in achieving scaled consensus for a *given* network, without changing its topology. That is, it is assumed that the graph is given or, more precisely, the coefficients a_{ij} and cannot be modified; in particular, edges cannot be added nor removed. The approach must consist in modifying the individual control inputs (1) so as to achieve (4), for any given set of weights $w \in R^N_{>0}$. To that end, we pose the ansatz

$$u_i = -\sum a_{ij}(\beta_{ij} x_i - \beta'_{ij} x_j), \quad i \in \{1, 2, \ldots, N\} \tag{8}$$

where $\beta_{ij}, \beta'_{ij} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, depend on w_i. To define them, we observe the following.

The closed-loop equations (1a)-(8) read

$$\dot{x}_i = -\sum_{j \neq i} a_{ij}(\beta_{ij} x_i - \beta'_{ij} x_j), \quad i \in \{1, 2, \ldots, N\}, \tag{9}$$

so to address the scaled consensus problem as one of stabilization, $x^* := [x_1 \cdot x_N^*]^\top$ in (6) must be a solution of (9). This is the case if

$$\beta_{ij} x_i^* = \beta'_{ij} x_j^* \qquad \forall i, j \in \{1, 2, \ldots, N\}. \tag{10}$$

Therefore, after (6) we see that (10) holds if and only if

$$\beta_{ij} w_i x_m = \beta'_{ij} w_j x_m \qquad \forall i, j \in \{1, 2, \ldots, N\}. \tag{11}$$

In turn, the latter holds if and only if, for any $k_{ij} > 0$, we define

$$\beta_{ij} := \frac{k_{ij}}{w_i}, \quad \beta'_{ij} := \frac{k_{ij}}{w_j} \qquad \forall i, j \in \{1, 2, \ldots, N\}. \tag{12}$$

Next, replacing (12) in the closed-loop equations (9) we obtain

$$\dot{x}_i = -\sum_{j \neq i} a_{ij} k_{ij} \Big(\frac{x_i}{w_i} - \frac{x_j}{w_j} \Big), \quad i \in \{1, 2, \ldots, N\}, \tag{13}$$

In multi-variable form we have

$$\dot{x} = -L_s x$$

$$L_s := \begin{bmatrix} \sum_{j=2}^N a_{1j}\beta_{1j} & -a_{12}\beta'_{12} & \cdots & -a_{1N}\beta'_{1N} \\ -a_{21}\beta'_{21} & \sum_{j \neq 2}^N a_{2j}\beta_{2j} & \cdots & -a_{2N}\beta'_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1}\beta'_{N1} & -a_{N2}\beta'_{N2} & \cdots & \sum_{j=1}^{N-1} a_{Nj}\beta'_{Nj} \end{bmatrix}.$$

The matrix L_s above has a structure similar to that of a Laplacian matrix, but L_s may not be considered as such because in the present context $\beta'_{ij} \neq \beta_{ij}$; consequently, L_s is not row stochastic (the sum of the elements of each row does not necessarily equal to zero). Nonetheless, the matrix L_s possess certain properties, inherited from the Laplacian L, that are useful to analyze (14). This is because the introduction of the individual weights β_{ij} and β'_{ij} on the measurements x_i and x_j in (8) do not modify the graph. More precisely, $\beta_{ij} > 0$ and $\beta'_{ij} > 0$ if and only if $a_{ij} > 0$. In the literature—see *e.g.,* [14]—matrices such as L_s are known as *generalized* or *scaled* Laplacian.

In general, scaled (by W) Laplacians may be defined as follows. Given a graph G with corresponding adjacency and Laplacian matrices be denoted A and $L \in \mathbb{R}^{N \times N}$

respectively. Then, defining $W = \mathrm{diag}[w_i]$, where $w_i > 0$ are given weights, the scaled Laplacian is given as

$$WLW^{-1} = \begin{bmatrix} l_{11} & -\frac{w_1}{w_2} l_{12} & \dots & -\frac{w_1}{w_n} l_{1N} \\ -\frac{w_2}{w_1} l_{21} & l_{22} & \dots & -\frac{w_2}{w_n} l_{2N} \\ \vdots & \vdots & & \vdots \\ -\frac{w_n}{w_1} l_{21} & -\frac{w_n}{w_2} l_{N2} & \dots & l_{NN} \end{bmatrix}, \quad (15)$$

and a simple inspection shows that the matrix above corresponds exactly to the matrix in (14a), under (12). That is,

$$L_s = WLW^{-1}, \quad (16)$$

which means that its off-diagonal entries are defined as

$$l_{s,ij} := -\frac{w_i}{w_j} l_{ij}. \quad (17)$$

For subsequent development, we remark the following.

Proposition 1 (Properties of the scaled Laplacian matrix):

(i) The matrices L and L_s have the same eigenvalues, so the matrix L_s has a unique null eigenvalue, $\lambda_1(L_s) = 0$, while all the others are positive;

(ii) the right and left eigenvectors of L_s, which are denoted by $v_{s,r}$, $v_{s,l}$, associated to $\lambda_1(L_s) = 0$ have the form

$$v_{S,r} = w, \quad v_{s,l} = W^{-1} v_l,$$

where v_l is the left eigenvector of L associated with $\lambda_1(L) = 0$;

(iii) if the underlying graph G is strongly connected, then $v_{s,l} \succ 0$, *i.e.,* all the components of $v_{s,l}$ are positive. □

Proof: Item (i): that the matrices L and L_s have exactly the same eigenvalues follows from the fact that the Jordan decomposition of L, yields $L = U\Lambda U^{-1}$. Hence, the matrix L_s can be written as $L_s = WU\Lambda U^{-1}W^{-1}$, so it admits the Jordan decomposition $L_s = U_1 \Lambda U_1^{-1}$, where $U_1 = WU$, where W is diagonal, hence full rank.

Item (ii): direct computations show that, for the right eigenvector, we have $L_s v_{s,r} = L_s w = \big[WLW^{-1}\big] w = WL\mathbf{1}_N = 0$. Then, for the left eigenvector, we have $v_{s,l}^\top L_s = v_l^\top W^{-1} L_s = v_l^\top W^{-1}\big[WLW^{-1}\big] = v_l^\top LW^{-1} = 0$.

Item (iii): that $v_{s,l} \succ 0$ follows directly from the fact that all the components of the vector v_l, which is the first left eigenvector of L, are positive, and the definition of $v_{s,l}$—see the proof of Item 2. ■

III. Main results

In what follows, we focus on multi-agent systems interconnected over a strongly connected graph. This assumption enables us to establish constructive proofs of scaled consensus for networks of first and second order systems, that is, based on Lyapunov's direct method to establish exponential stability of the synchronization manifold (5).

With that under consideration, the Laplacian L, whose elements are defined in (3) is an irreducible singular M-matrix and its left and right eigenvectors associated with the unique eigenvalue $\lambda = 0$ satisfy $v_r \succ 0$ and $v_l \succ 0$ respectively. Furthermore, without loss of generality, it is further assumed that v_l and v_r are normalized, that is $v_l^\top v_r = 1$.

Then, for Laplacians of strongly directed graphs we have the following.

Lemma 1: [15] Let L be the Laplacian matrix corresponding to a strongly connected digraph G, let $v_l \in R^N$ denote the left eigenvector associated with the unique eigenvalue $\lambda(L) = 0$, and define $W := \mathrm{diag}[v_{\ell i}]$. Then, $\lambda_2\big(WL + L^\top W\big) > 0$. □

Lemma 2: [16] Consider a a strongly connected graph G with associated Laplacian matrix L. Let $v_l := [v_{l1} \ \cdots \ v_{lN}]^\top$ and $v_r := [v_{r1} \ \cdots \ v_{rN}]^\top$ denote its left and right eigenvectors associated with the unique eigenvalue $\lambda_1(L) = 0$. Then, the matrices

$$P := \mathrm{diag}[p_i]; \quad p_i := \frac{v_{li}}{v_{ri}}, \quad i \leq N$$
$$Q := PL + L^\top P$$

satisfy: $P > 0$ and $Q \geq 0$. □

A. First-order systems

Proposition 2: Consider the system (14), where L_s is a scaled Laplacian matrix. If the underlying graph is strongly connected, then the synchronization manifold $\mathcal{S}$ defined in (5) is globally exponentially stable. Consequently, the synchronization errors, defined as

$$\mathbf{e}_i = \mathbf{x}_i - w_i \mathbf{x}_m, \quad i \leq N, \quad (18)$$

where $\mathbf{x}_m = v_{s,l}^\top \mathbf{x}$, converge to zero exponentially. □

Proof: Consider the Lyapunov function

$$V(\mathbf{e}) = \mathbf{e}^\top P \mathbf{e}, \quad (19)$$

where the matrix P is defined as

$$P := \mathrm{diag}\left[\frac{(v_{s,l})_i}{(v_{s,r})_i}\right] = \mathrm{diag}\left[\frac{(v_{s,l})_i}{w_i}\right]. \quad (20)$$

Notice that since the underlying graph is strongly connected, all the entries of the vector v_{l1} are positive. Hence, the matrix P is well defined for strongly connected graphs. We denote

$$p_M := \max_{i \leq N} \left\{ \frac{(v_{s,l})_i}{w_i} \right\}, \quad p_m := \min_{i \leq N} \left\{ \frac{(v_{s,l})_i}{w_i} \right\}. \quad (21)$$

Then, since the scaled Laplacian matrix is defined as in (15) for any Laplacian matrix L, from Claim 1, it follows that $v_{s,ri} = w_i$ and $v_{s,li} = v_{li}(L)/w_i$. Then, the matrix P takes the form

$$P = \mathrm{diag}\left[\frac{v_{li}(L)}{w_i^2}\right] = \mathrm{diag}[v_l(L)]W^{-2}. \quad (22)$$

Next, we evaluate the derivative of $V(\mathbf{e})$ along solutions of $\dot{\mathbf{e}} = -L_s \mathbf{e}$. We obtain

$$\dot{V}(e) = -\mathbf{e}^\top L_p \mathbf{e}, \quad (23)$$

where

$$L_p = PL_s + L_s^\top P. \quad (24)$$

Then, after (16) and (22), we have

$$L_p = W^{-1}\Big[\operatorname{diag}[v_l(L)]L + L^\top \operatorname{diag}[v_l(L)]\Big]W^{-1}.$$

We stress that since the matrix L is a standard Laplacian matrix, $Q := \operatorname{diag}[v_l(L)]L + L^\top \operatorname{diag}[v_l(L)] \geq 0$—see Lemma 2. Moreover, Q has only one zero eigenvalue, which corresponds to $\lambda_1(L)$, and the corresponding eigenvectors $v_l(Q) = v_r(Q) = \mathbf{1}_N$. Then, we see that $L_p = W^{-1}QW^{-1}$ has only one zero eigenvalue with corresponding eigenvectors $v_{s,r} = v_{s,l} = w$, and all the others are positive. After Lemma 1, $\dot V(e) \leq -\lambda_2(L_p)|\boldsymbol{e}|^2$. Global exponential stability of $\{\boldsymbol{e} = 0\}$ and, equivalently of the manifold $\mathcal{S}$ in (5), follows from the latter and (23). ■

B. Illustrative example

For illustration, we present below, the results of a simple simulation of a network of five first-order systems interconnected over a directed-ring topology. In the numerical test, the initial conditions are set to: $x_1(0) = 2.5$, $x_2(0) = 1$, $x_3(0) = -0.5$, $x_4(0) = 1.4$, $x_5(0) = -1.2$ with respective weights $w_1 = 0.1$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.3$, and $w_5 = 0.25$. As expected, all the systems converge to a consensual value that equals to the sum of the initial conditions, factored by their respective weights.

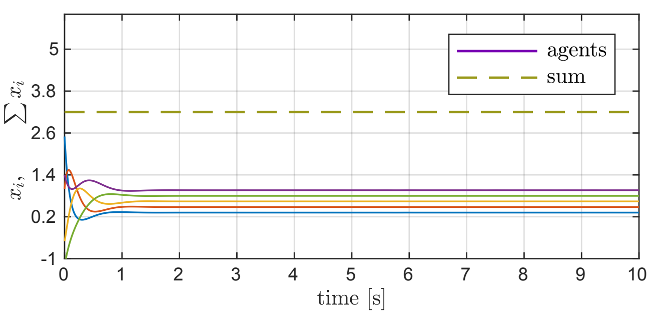

Fig. 1. Leaderless scenario of scaled consensus

C. Second-order systems

Consider a network of second-order systems interconnected over a directed strongly connected graph, as follows:

$$\dot x_{i1} = x_{i2}, \quad i \in \{1, 2, \ldots, N\} \tag{25a}$$
$$\dot x_{i2} = u_i, \tag{25b}$$
$$u_i = -\sum_{j\neq i} a_{ij}(\beta_{ij}x_i - \beta'_{ij}x_j) - \gamma x_{i2}, \tag{25c}$$

where $a_{ij} \geq 0$, $\gamma > 0$, and β_{ij} and β'_{ij} are defined in (12). In multi-variable form, the closed-loop system is:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2, \tag{26a}$$
$$\dot{\mathbf{x}}_2 = -L_s\mathbf{x}_1 - \gamma\mathbf{x}_2, \tag{26b}$$

where L_s is a scaled Laplacian matrix, as defined by (15)-(16); its right eigenvector corresponding to the unique null eigenvalue $\lambda_1(L_s) = 0$, is given by $v_r^\top = [w_1, \ldots, w_n]$.

Proposition 3: Consider the system (26), where L_s is a scaled Laplacian matrix corresponding to a strongly connected graph. Then, the synchronization manifold—*cf.* (5),

$$S := \left\{\boldsymbol{x} \in \mathbb{R}^N \,:\, \frac{x_{11}}{w_1} = \frac{x_{21}}{w_2} = \cdots = \frac{x_{N1}}{w_N} = \text{const}\right\}, \tag{27}$$

is globally exponentially stable and, in particular, the synchronization errors

$$\mathbf{e}_{1i} = \mathbf{x}_{1i} - w_i\mathbf{x}_{m1}, \tag{28a}$$
$$\mathbf{e}_{2i} = \mathbf{x}_{2i} - w_i\mathbf{x}_{m2}, \tag{28b}$$

where $\mathbf{x}_{mi} = v_{s,l}^\top\mathbf{x}_i$ with $i = 1, 2$, converge to zero exponentially. □

Proof: The dynamics of the synchronization errors $\mathbf{e}_1$, $\mathbf{e}_2$ in (28) is given by

$$\dot{\mathbf{e}}_1 = \mathbf{e}_2, \tag{29a}$$
$$\dot{\mathbf{e}}_2 = -L_s\mathbf{e}_1 - \gamma\mathbf{e}_2. \tag{29b}$$

Then, for this system consider a Lyapunov function defined as

$$V(\mathbf{e}_1, \mathbf{e}_2) = \frac{1}{2}\mathbf{e}_1^\top L_p\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2^\top P\mathbf{e}_2 + \epsilon\mathbf{e}_1 P\mathbf{e}_2, \tag{30}$$

where matrices L_p and P are defined in (20) and (24)

The Lyapunov function defined in (30) is positive definite and radially unbounded if

$$\epsilon < \frac{\sqrt{p_m\lambda_2(L_p)}}{p_M}. \tag{31}$$

Indeed,

$$V(\mathbf{e}_1, \mathbf{e}_2) \geq \frac{1}{2}\lambda_2(L_p)|\mathbf{e}_1|^2 + \frac{1}{2}p_m|\mathbf{e}_2|^2 - \epsilon p_M|\mathbf{e}_1|\,|\mathbf{e}_2|, \tag{32}$$

that is,

$$\begin{aligned} V(\mathbf{e}_1, \mathbf{e}_2) \geq\ & \frac{1}{4}\lambda_2(L_p)|\mathbf{e}_1|^2 + \frac{1}{4}p_m|\mathbf{e}_2|^2 \\ & + \frac{1}{2}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix}^\top \begin{bmatrix}\lambda_2(L_p)/2 & \epsilon p_M\\ \epsilon p_M & p_m/2\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix}, \end{aligned}$$

and the matrix above is positive definite if

$$\epsilon < \frac{\sqrt{\lambda_2(L_p)p_m}}{2p_M}.$$

Next, we evaluate the derivative of $V(\mathbf{e}_1, \mathbf{e}_2)$ along the trajectories of (29). This is

$$\begin{aligned} \dot V(\mathbf{e}_1, \mathbf{e}_2) =\ & \mathbf{e}_1^\top\left[L_p - PL_s - L_sP\right]\mathbf{e}_2 - \epsilon\mathbf{e}_1^\top PL_s\mathbf{e}_1 \\ & - \mathbf{e}_2^\top\left[\frac{1}{2\epsilon}L_p + (\gamma - \epsilon)P\right]\mathbf{e}_2 - \epsilon\gamma\mathbf{e}_1^\top P\mathbf{e}_2, \end{aligned}$$

and, in view of (24), we obtain

$$\begin{aligned} \dot V(\mathbf{e}_1, \mathbf{e}_2) \leq\ & -\frac{\epsilon}{2}\lambda_2(L_p)|\mathbf{e}_1|^2 + \epsilon\gamma p_M|\mathbf{e}_1|\,|\mathbf{e}_2| \\ & - \left[\gamma p_m + \frac{1}{2\epsilon}\lambda_2(L_p) - \epsilon p_M\right]\ |\mathbf{e}_2|^2. \end{aligned}$$

Hence, we see that $\dot V(\mathbf{e}_1, \mathbf{e}_2) \leq 0$ is negative definite if

$$\sqrt{\lambda_2(L_p)/2p_M} > \epsilon \tag{33}$$

and

$$-\frac{\epsilon}{2}\lambda_2(L_p)|\mathbf{e}_1|^2 - \gamma p_m|\mathbf{e}_2|^2 + \epsilon\gamma p_M|\mathbf{e}_1|\,|\mathbf{e}_2| < 0, \quad (34)$$

which in turn holds if

$$\epsilon < \frac{2\lambda_2(L_p)\,p_m}{\gamma p_M^2}.$$

Thus, the Lyapunov function $V(\mathbf{e}_1, \mathbf{e}_2)$ defined in (30) is positive definite and radially unbounded provided that

$$\epsilon < \min\left\{\frac{\sqrt{p_m\lambda_2(L_p)}}{p_M},\ \frac{\sqrt{\lambda_2(L_p)}}{2p_M},\ \frac{2\lambda_2(L_p)\,p_m}{\gamma p_M^2}\right\}.$$

The latter, however, holds without any restriction neither on the control gain γ nor on the network's topology. ■

D. Scaled leader-follower consensus

We consider now a scenario of leader-follower consensus for second-order systems that consists in making a given network of N agents with states $x_i \in \mathbb{R}^2$, with $i \in \{1, 2, \ldots, N\}$ interconnected over a strongly connected graph G, follow a leader. The state of the latter is denoted by $x_\ell \in \mathbb{R}^2$. That is, in particular, the network's graph contains a rooted spanning tree and the leader node acts upon the root node of the followers network. More precisely, we impose the following.

Assumption 1: The digraph G corresponding to the followers network contains a spanning tree, and at least the root node receives information from the leader node, that is, the input to the ith node includes a term of the form

$$v_i := -a_{i\ell}(x_{i1} - x_{\ell 1}), \quad a_{i\ell} \geq 0. \quad (35)$$

The interconnection term $a_{i\ell} > 0$ if the leader node acts on the ith node and $a_{i\ell} = 0$ otherwise. •

Remark 2: As per Assumption 1, $a_{1\ell} > 0$. •

Motivated by scenarii that are common in electrical networks, we assume that dynamics of the leader node is of the form

$$\dot{x}_{\ell 1} = x_{\ell 2} \quad (36a)$$

$$\dot{x}_{\ell 2} = -\alpha(x_{\ell 1} - c^*) - \beta x_{\ell 2}, \quad (36b)$$

where $\alpha,\ \beta, c^* > 0$. That is, it is assumed that the leader node is controlled to make its first state variable converge to a preimposed reference point c^*. Clearly, the equilibrium point $\{(x_{\ell 1} = c^*, x_{\ell 2}) = (0, 0)\}$ is globally exponentially stable. The leader-node dynamics can be viewed as those of an electrical bus whose operating point must be regulated despite the presence of a connected load. A set of interconnected generator nodes (followers) contributes to this regulation by injecting controlled power into the bus, each providing a prescribed share of the total demand. The coordinated action of all generators ensures that the aggregate injected power balances the load, thereby achieving power (or current) sharing as a collective control objective.

The follower nodes use consensus controllers similar to that defined in (25c), but with an additional term of the form given in (35). More precisely, we have the following.

Proposition 4: Consider the systems (25) with u_i redefined as

$$u_i = -\sum_{j\neq i} a_{ij}(\beta_{ij}x_i - \beta'_{ij}x_j) - \gamma x_{i2} - a_{i\ell}(x_{i1} - x_{\ell 1}), \quad (37)$$

where $a_{ij} \geq 0,\ a_{i\ell} \geq 0$, for all $i,\ j \leq N,\ a_{1\ell},\ \gamma > 0$, and $x_{\ell 1}$ is defined in (36). Then, the synchronization manifold S in (5) is globally exponentially stable and, moreover, $x_i \to w_i c^*$, $\dot{x}_i \to 0$ for all $i \leq N$. □

Proof: We start by writing the overall closed-loop network system in multi-variable form, that is,

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2, \quad (38a)$$

$$\dot{\mathbf{x}}_2 = \mathrm{diag}[w]\mathbf{d}_{\ell 1}x_{\ell 1} - \big[L_s + \mathrm{diag}[\mathbf{d}_{\ell 1}]\big]\mathbf{x}_1 - \gamma\mathbf{x}_2, \quad (38b)$$

where the vector $\mathbf{d}_{\ell 1} \in \mathbb{R}^n_{\geq 0}$ contains the coefficients $a_{i\ell}$ defined above.

Remark 3: Since the network of the followers is strongly connected, then to ensure that the overall network has a spanning tree it is enough to have at least one element of the vector $\mathbf{d}_{\ell 1} \neq 0$. The latter holds under Assumption 1. •

For the sequel, we define $\bar{\mathbf{x}}_1 := \big[x_{\ell 1}\ \mathbf{x}_1^\top\big]^\top \in \mathbb{R}^{N+1}$ and $\bar{\mathbf{x}}_2 := \big[x_{\ell 2}\ \mathbf{x}_2^\top\big]^\top \in \mathbb{R}^{N+1}$, to denote the overall state of the network, including the leader node's. In these coordinates, we write the dynamics of the systems (36)-(38) together, as

$$\dot{\bar{\mathbf{x}}}_1 = \bar{\mathbf{x}}_2,$$

$$\dot{\bar{\mathbf{x}}}_2 = \begin{bmatrix} -\alpha(\bar{\mathbf{x}}_{11} - c^*) - \beta\bar{\mathbf{x}}_{22} \\ 0_{N\times 1}\end{bmatrix} - \bar{L}_s\bar{\mathbf{x}}_1 - \gamma I_f\mathbf{x}_2,$$

where the matrices $I_f \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\bar{L}_s \in \mathbb{R}^{(N+1)\times(N+1)}$ are defined as

$$I_f = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}, \quad \bar{L}_s = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathrm{diag}[w]\mathbf{d}_{\ell 1} & -L_s - \mathrm{diag}[\mathbf{d}_{\ell 1}] \end{bmatrix}.$$

The matrix $\bar{L}_s$ defined above has one zero eigenvalue $\lambda_1(\bar{L}_s) = \lambda_1(L_s) = 0$, and the corresponding eigenvectors $\bar{v}_{r1} = [1\ w^\top]^\top$ and $\bar{v}_{\ell 1} = [1\ 0\ \cdots\ 0]^\top$.

As before we can rewrite the system in terms of errors coordinates that take the form

$$\bar{\mathbf{e}}_1 = \bar{\mathbf{x}}_1 - \bar{v}_{r1}v_{\ell 1}^\top\bar{\mathbf{x}}_1 = \bar{\mathbf{x}}_1 - \bar{v}_{r1}x_{\ell 1}$$

$$\bar{\mathbf{e}}_2 = \bar{\mathbf{x}}_2 - \bar{w}\bar{x}_{\ell 2},$$

that is, $\bar{e}_1,\ \bar{e}_2 \in \mathbb{R}^{N+1}$. Notice that with this defined errors, we have that $\bar{\mathbf{e}}_{1,1} \equiv 0$ and $\bar{\mathbf{e}}_{2,1} \equiv 0$. Hence we can write dynamics in terms of the coordinates

$$\mathbf{e}_1 = \mathbf{x}_1 - wx_{\ell 1}$$

$$\mathbf{e}_2 = \mathbf{x}_2 - wx_{\ell 2},$$

that is, $\bar{e}_1,\ \bar{e}_2 \in \mathbb{R}^N$. In these coordinates, the synchronization error dynamics takes the form

$$\dot{\mathbf{e}}_1 = \mathbf{e}_2$$

$$\dot{\mathbf{e}}_2 = -\big[L_s + \mathrm{diag}[\mathbf{d}_{\ell 1}]\big]\mathbf{e}_1 - \gamma\mathbf{e}_2 - c_1(x_{\ell 1} - c^*)w - c_2x_{\ell 2}w.$$

Now, since the origin is globally exponentially stable for the dynamics of the leader node, (38) and the latter is independent of that of the followers, we can employ a cascades argument: global exponential stability of the origin $(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2) = (0, 0)$ for the overall system follows if $(\mathbf{e}_1, \mathbf{e}_2) = (0, 0)$ is also globally exponentially stable for the system

$$\dot{\mathbf{e}}_1 = \mathbf{e}_2 \tag{39a}$$
$$\dot{\mathbf{e}}_2 = -\big[L_s + \mathrm{diag}[\mathbf{d}_{\ell 1}]\big]\mathbf{e}_1 - \gamma \mathbf{e}_2. \tag{39b}$$

To see this, we stress that, by assumption, the non-negative vector $\mathbf{d}_{\ell 1}$ has at least one non-zero element, then using Lemma 3 from the Appendix, and arguments similar to those used in Theorem 3, we can show the the origin is globally exponentially stable for the system (39). ■

E. Illustrative example

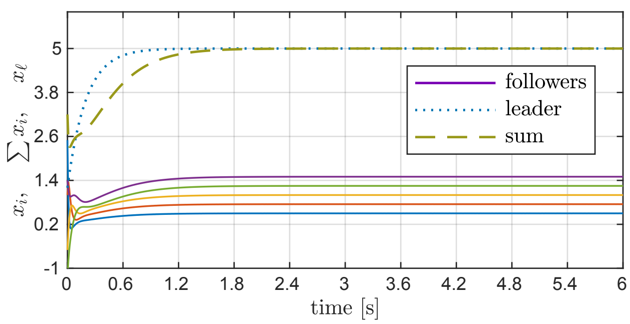

Fig. 2. Leader follower scenario of scaled consensus

For illustration, we present next the results of a simulation involving a network of five first-order systems interconnected over a directed-ring topology, to which a leader node is added as described above. In the numerical test, the initial conditions of the follower nodes are set to the same values as in the previous example, *i.e.,* $x_1(0) = 2.5$, $x_2(0) = 1$, $x_3(0) = -0.5$, $x_4(0) = 1.4$, $x_5(0) = -1.2$ and with the same respective weights $w_1 = 0.1$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.3$, and $w_5 = 0.25$. However, a leader node is controlled to stabilize at the set point $c^* := 5$. As before, all the systems converge to scaled consensual values; the sum of their steady states equals to the desired value (imposed by the leader node). This scenario is illustrative of an electrical network in which a group of generators are meant to contribute with a fraction of the demanded power (represented by the leader node's reference set-point).

IV. Conclusions

We presented some results on scaled consensus that remain somewhat preliminary since they are limited to the problem of classical consensus over static-topology networks and are restricted to first- and second-order driftless systems. However, we believe that our results are useful in as much as they provide strict Lyapunov functions to establish strong stability properties. Therefore, they may be the starting point to analyze problems inspired from domains such as Electrical networks engineering and Robotics, where problems of scaled consensus of multi-agent systems arise naturally. Current research is carried out in these directions.

Appendix

The following is a paraphrasis of [8, Lemma 14].

Lemma 3: Let L be the Laplacian matrix corresponding to a connected digraph containing a spanning tree. Under Assumption 1, the matrix $L + A_\ell$, where $A_\ell := \mathrm{diag}[a_{i\ell}] \succeq 0$, is non-singular. Furthermore, define

$$q := [L + A_\ell]^{-1}\mathbf{1}_N =: [q_1 \ \cdots \ q_N]^\top$$
$$P := \mathrm{diag}[p_i], \quad p_i := \left[\frac{1}{q_i}\right]$$
$$Q := P[L + A_\ell] + [L + A_\ell]^\top P.$$

Then, $P > 0$ and $Q > 0$. □

References

[1] A. Cristofaro and M. Mattioni, "Multiconsensus control of homogeneous LTI hybrid systems under time-driven jumps," in *IEEE 61st Conference on Decision and Control*, pp. 316–321, IEEE, 2022.

[2] S. Monaco and L. R. Celsi, "On multi-consensus and almost equitable graph partitions," *Automatica*, vol. 103, pp. 53–61, 2019.

[3] L. V. Gambuzza and M. Frasca, "Distributed control of multiconsensus," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2032–2044, 2020.

[4] S. Roy, "Scaled consensus," *Automatica*, vol. 51, pp. 259–262, 2015.

[5] C. Tomaselli, L. V. Gambuzza, F. Sorrentino, and M. Frasca, "Control of multiconsensus in multi-agent systems based on eigenvector centrality," *Automatica*, vol. 164, p. 111638, 2024.

[6] Z. Wang, L. Wang, H. Zhang, and Q. Chen, "A graph based formation control of nonholonomic wheeled robots using a novel edge-weight function," in *2017 IEEE International Conference on Systems, Man, and Cybernetics (SMC)*, (Banff, AB), pp. 1477–1481, IEEE, Oct. 2017.

[7] F. Dörfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 60, no. 1, pp. 150–163, 2012.

[8] H. Zhang, F. L. Lewis, and Z. Qu, "Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3026–3041, 2011.

[9] D. Meng and Y. Jia, "Scaled consensus problems on switching networks," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1664–1669, 2015.

[10] J. Yu and Y. Shi, "Scaled group consensus in multiagent systems with first/second-order continuous dynamics," *IEEE Transactions on Cybernetics*, vol. 48, no. 8, pp. 2259–2271, 2017.

[11] Q. Wang, "Scaled consensus of multi-agent systems with output saturation," *Journal of the Franklin Institute*, vol. 354, no. 14, pp. 6190–6199, 2017.

[12] N. Alvarez-Jarquin, A. Loría, and J. L. Avila, "Consensus under switching spanning-tree topologies and persistently exciting interconnections," in *Proc. IEEE American Control Conference*, (Milwaukee, WI, USA), pp. 4578–4583, 2018.

[13] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. London, U.K.: Springer verlag, 2008.

[14] G. F. Royle and C. Godsil, *Algebraic graph theory*, vol. 207. New York: Springer, 2001.

[15] C. W. Wu, "On Rayleigh-Ritz ratios of a generalized Laplacian matrix of directed graphs," *Linear algebra and its applications*, vol. 402, pp. 207–227, 2005.

[16] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proceedings of the 2005, American Control Conference, 2005.*, pp. 1859–1864, IEEE, 2005.