Refined Unknown Input Observer Design for Time-Delay Systems with Uncertainties

Chien-Shu Hsieh

***Abstract*— This paper examines the problem of unknown input estimation for a class of systems subject to known time-varying delays in their state variables and additive, uncertain, nonlinear disturbances. Unlike traditional designs that rely on state augmentation to construct simultaneous state and input observers, the proposed approach employs a reformed system transformation to reduce the computational complexity associated with refined unknown input observer (RUIO) design. Sufficient conditions are derived to ensure the solvability of the RUIO design, enabling the construction of a reduced-order RUIO design. This paper further addresses compact issues from the perspective of finite-precision implementation by investigating the compact design of system matrices. A simulation example is presented to illustrate the design procedure and to verify the effectiveness of the proposed approach.**

***Index Terms*— Reduced-order observer, system reformation, refined unknown input observer, finite-precision implementation.**

I. INTRODUCTION

Simultaneous input and state estimation (SISE) is a useful approach for estimating the states and unknown inputs of time-invariant systems as well as those of uncertain/nonlinear systems. This problem is crucial because it arises in applications involving fault detection and isolation [1] and in situations where measurement of unknown inputs is either too expensive [2] or physically infeasible [3]. In chaotic secure communication, an effective method for enhancing encryption is the master-slave synchronization approach based on chaotic systems. The underlying idea is to induce specific chaotic behavior in the encryption system and to design a stable decryption system that synchronizes with it [4], [5], [6]. Nevertheless, this approach focuses on simultaneous chaotic synchronization and message recovery, which may not always necessary [7], [8]. An alternative SISE (ASISE) technique, originally proposed in [9] for solving the delay-estimable unknown input filtering problem, is adopted to facilitate the design of the decryption system. The main advantages of this approach are that it does not require derivatives of output measurements and focuses solely on message recovery, thereby reducing the computational complexity of the receiver system.

Time-delay state estimation is frequently encountered in many practical engineering problems (see [10] for related references). It is well known that a key distinction between systems with and without time delay lies in their dynamics, which are infinite-dimensional rather than finite-dimensional dynamics, as in many delay-free systems [11]. As a result, uncertain time-delay chaotic systems can potentially enhance the encryption performance of secure communication systems. The main objective of this paper is to further investigate the extension of ASISE to uncertain time-delay and uncertain chaotic-based encryption to improve encryption performance. A critical step toward this objective is the study of unknown input estimation for time-delay systems subject to uncertainties.

One convenient approach to achieving SISE for uncertain time-delay systems is based on reduced-order observer (ROO) design [12]. In this approach, a descriptor system incorporating both system states and unknown inputs is first constructed. A reduced-order observer with unknown system matrices is then assumed to ensure asymptotic convergence to the descriptor state. Subsequently, a set of matrix restriction conditions is derived and solved algebraically, yielding an linear matrix inequality (LMI) formulation whose solution guarantees asymptotic convergence of the estimation error. However, the method proposed in [12] has limited applicability, as it is restricted to systems with direct feedthrough. In addition, the reliance on solving LMIs makes the method computationally demanding. The main aim of this paper is to develop an alternative reduced-order observer design that overcomes these limitations. The proposed method employs a reformed system transformation technique [9] to reduce the computational complexity of the resulting obtained refined unknown input observer (RUIO). A reduced-order version of the RUIO, named as the RRUIO, is also developed. This paper further examines implementation issues associated with the RRUIO, with the goal of preserving filtering performance under degraded computing conditions [13]. It is showed that the RRUIO yields more compact system matrices than those of the ROO.

The remainder of this paper is organized as follows. Section II presents the problem formulation and reviews the ROO design. Section III describes the proposed RUIO and RRUIO designs for uncertain time-delay systems. Specifically, Section III.A derives the proposed RUIO design. Section III.B derives the proposed RRUIO design. Section III.C provides a comparison between the ROO and the RRUIO designs, and Section III.D discusses finite-precision consideration for both the RRUIO and ROO. Section IV presents a simulation example to illustrate the design procedures and demonstrate the effectiveness of the proposed results. Finally, Section V presents the conclusions.

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II. PROBLEM FORMULATION

This study considers the following time-delay continuous-time system with uncertainties:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + Bu(t) + Wf(t,x) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^q$, and $y(t) \in R^p$ denote the state vector, the unknown input vector to be estimated, and the measured output vector, respectively. Matrices A, A_d, B, W, C, and D are real constants of appropriate dimensions. The time-varying time delay $\tau(t)$ is subject to $0 \leq \tau(t) \leq \tau_u$ and $\dot{\tau}(t) \leq \alpha < 1$ and the unknown disturbance $f(t,x) \in R^d$, which is not required to be estimated, represents system uncertainties, nonlinearities, and time-varying terms. The matrix W, assumed to have full-column rank, characterizes the distribution of the disturbance within the system dynamics. It is assumed that the pair $(A+A_d, C)$ is detectable, and that the unknown input vector $u(t)$ is estimable. The objective of this paper is to determine the unknown input vector $u(t)$ without explicitly estimating the system state.

A common approach to simultaneously estimating the state and unknown input vector begins by defining an augmented state vector $z(t)$, where $z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$. Under this formulation, the time-delay system (1) can be rewritten as:

$$\begin{cases} E\dot{z}(t) = \bar{A}z(t) + \bar{A}_d z(t-\tau(t)) + Wf(t,x) \\ y(t) = \bar{C}z(t) \end{cases} \tag{2}$$

where $E = \begin{bmatrix} I & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & B \end{bmatrix}$, $\bar{A}_d = \begin{bmatrix} A_d & 0 \end{bmatrix}$, and $\bar{C} = \begin{bmatrix} C & D \end{bmatrix}$. Then, subject to the following rank condition:

$$rank \begin{bmatrix} D & CW \end{bmatrix} = q + d \tag{3}$$

the following reduced-order observer (ROO) [12] can be applied to address the SISE problem (2):

$$\begin{cases} \dot{\omega}(t) = N\omega(t) + N_d\omega(t-\tau(t)) + Ly(t) \\ \qquad + L_d y(t-\tau(t)) \\ \hat{z}(t) = M\omega(t) + Fy(t) \end{cases} \tag{4}$$

where matrices N, N_d, L, L_d, M, and F satisfy the following conditions:

$$\begin{cases} NTE + L\bar{C} - T\bar{A} = 0 \\ N_d TE + L_d\bar{C} - T\bar{A}_d = 0 \\ TW = 0 \\ \begin{bmatrix} M & F \end{bmatrix} \begin{bmatrix} TE \\ \bar{C} \end{bmatrix} = I_{n+m} \end{cases} \tag{5}$$

Once these conditions are satisfied, the unknown input vector $u(t)$ can be obtained as follows:

$$u(t) = \begin{bmatrix} 0 & I \end{bmatrix} (M\omega(t) + Fy(t)) \tag{6}$$

Nevertheless, the above approach has several drawbacks, which are summarized as follows:

(1) As shown in [12], the solvability of the equivalent system (2) can be achieved only when the rank condition (3) is satisfied, which requires the matrix D to be of full-column rank.
(2) A key step solving problem (4), (5) is the determination of matrix T, whose value is obtained by solving an LMI solution (see Section III.C for details). This procedure can be tedious to use.
(3) The resulting reduced-order observer is not expressed in its most compact form, which may lead to implementation issues.

Thus, the main objective of this paper is to address an alternative reduced-order observer design that overcomes the above limitations through the following contributions:

(1) A generalized rank condition (3) is proposed to address the more general case in which the matrix D may have an arbitrary rank. Furthermore, when matrix D is not of full-column rank, the estimation of the unknown input vector requires derivatives of the output measurements. Thus, a new approach that avoids output differentiation is introduced.
(2) A direct simple method based on unknown input filtering is proposed to estimate the unknown input vector without solely relying on state estimation, thereby addressing the unknown input estimation problem in the original system (1). In this approach, the resulting solution is unique and can be derived without solving an LMI.
(3) The resulting unknown input observer is expressed in a more compact form than that presented in [12]. This is achieved by transforming the original system into an alternative reformed system, following the approach suggested in [7].

III. MAIN RESULTS

A. Reformed Unknown Input Observer Design

First, the reformed state transformation (RST) is defined to reflect the effect of un-estimable unknown inputs on the outputs. To achieve this, matrix D is given as its full-rank factorization: $D = D_2\tilde{D}$. Let $\bar{D}$ be a full-row rank matrix such that $det \begin{bmatrix} \bar{D} \\ \tilde{D} \end{bmatrix} \neq 0$. Defining the following unknown input transformation:

$$u(t) = \begin{bmatrix} \bar{D} \\ \tilde{D} \end{bmatrix}^{-1} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{7}$$

the original system (1) can be conveniently expressed as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + B_1 u_1(t) \\ \qquad + B_2 u_2(t) + Wf(t,x) \\ y(t) = Cx(t) + D_2 u_2(t) \end{cases} \tag{8}$$

where $B_1 = BU$ and $B_2 = BV$. Based on (8), the RST can be defined, to reflect u_1 on y, as:

$$\tilde{x}(t) = x(t) - B_1 u_1^*(t) - Wf^*(t,x) \tag{9}$$

where the notation $s^*(t)$ denotes $\frac{d}{dt}s^*(t) = s(t)$. Substituting (9) into (8) yields the following reformed system (RS):

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + A_d\tilde{x}(t-\tau(t)) + B_2u_2(t) + AB_1u_1^*(t) \\ \quad +AWf^*(t,x) + A_dB_1u_1^*(t-\tau(t)) \\ \quad +A_dWf^*(t-\tau(t),x) \\ y(t) = C\tilde{x}(t) + CB_1u_1^*(t) + CWf^*(t,x) + D_2u_2(t) \end{cases}$$

which can be rewritten as:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + A_d\tilde{x}(t-\tau(t)) + Gd(t) \\ \quad +AWf^*(t,x) + G_dd(t-\tau(t)) \\ \quad +A_dWf^*(t-\tau(t),x) \\ y(t) = C\tilde{x}(t) + Hd(t) + CWf^*(t,x) \end{cases} \tag{10}$$

where $d(t) = \begin{bmatrix} u_1^*(t) \\ u_2(t) \end{bmatrix}$, $G = \begin{bmatrix} AB_1 & B_2 \end{bmatrix}$, $G_d = \begin{bmatrix} A_dB_1 & 0 \end{bmatrix}$, and $H = \begin{bmatrix} CB_1 & D_2 \end{bmatrix}$.

Next, the disturbances are eliminated. From (10), the disturbances are estimated as:

$$f^*(t,x) = (CW)^+(y(t) - C\tilde{x}(t) - Hd(t)) \tag{11}$$

where CW is assumed to be a full-column matrix and M^+ denotes the pseudo-inverse of matrix M. Substituting this expression into (10) leads to:

$$\begin{cases} \dot{\tilde{x}}(t) = \breve{A}\tilde{x}(t) + \breve{A}_d\tilde{x}(t-\tau(t)) + \breve{G}d(t) \\ \quad +\breve{G}_dd(t-\tau(t)) + AW(CW)^+y(t) \\ \quad +A_dW(CW)^+y(t-\tau(t)) \\ \tilde{T}y(t) = \tilde{T}C\tilde{x}(t) + \tilde{T}Hd(t) \end{cases} \tag{12}$$

where

$$\begin{cases} \breve{A} = A(I - W(CW)^+C) \\ \breve{A}_d = A_d(I - W(CW)^+C) \\ \breve{G} = G - AW(CW)^+H \\ \breve{G}_d = G_d - A_dW(CW)^+H \\ \tilde{T}CW = 0 \end{cases} \tag{13}$$

Then, from (12) and assuming that $\tilde{T}H$ be a full-column matrix, the value of $d(t)$ can be obtained as:

$$d(t) = (\tilde{T}H)^+\tilde{T}(y(t) - C\tilde{x}(t)) \tag{14}$$

where matrix $\tilde{T}$ is determined, using the last constraint in (13), as:

$$\tilde{T} = \gamma(I - CW(CW)^+) \tag{15}$$

where matrix γ is chosen such that $\tilde{T}$ has a full-row rank. Substituting (14) into the reformed state dynamics in (12) yields the unknown-input-decoupled reformed state dynamics:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{A}_d\tilde{x}(t-\tau(t)) + \tilde{B}y(t) + \tilde{B}_dy(t-\tau(t)) \tag{16}$$

where

$$\begin{cases} \tilde{A} = \breve{A} - \breve{G}(\tilde{T}H)^+\tilde{T}C \\ \tilde{A}_d = \breve{A}_d - \breve{G}_d(\tilde{T}H)^+\tilde{T}C \\ \tilde{B} = AW(CW)^+ + \breve{G}(\tilde{T}H)^+\tilde{T} \\ \tilde{B}_d = A_dW(CW)^+ + \breve{G}_d(\tilde{T}H)^+\tilde{T} \end{cases} \tag{17}$$

Thus, Eqs. (14)-(17) constitute the reformed unknown input observer (RUIO).

Finally, we note that the above RUIO has the following advantages compared to those of the ROO:

(1) From (11) and (14), the existence condition of the RUIO is given by:

$$rank[CW] = d, \quad rank[\tilde{T}H] = q \tag{18}$$

which relaxes the limitation imposed by the rank condition (3).

(2) In the derivation of the RUIO, no specific matrix solution is sought. Instead, only unknown signals are determined. Thus, no LMI solution is required.

(3) The system matrices of the RUIO are more compact than those of ROO, which may have some advantages from the perspective of finite-precision implementation (see Section IV.B for details). This improvement results from avoiding a complex choice of the matrix T (35).

B. Reduced-order Reformed Unknown Input Observer Design

In this subsection, we show that the above proposed RUIO can be implemented more efficiently as below. To simplify the discussions, Eqs. (14) and (16) are rewritten as follows:

$$\begin{aligned} d(t) &= K(y(t) - C\tilde{x}(t)), \quad K = (\tilde{T}H)^+\tilde{T} && (19) \\ \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{A}_d\tilde{x}(t-\tau(t)) + \tilde{B}y(t) \\ &\quad +\tilde{B}_dy(t-\tau(t)) && (20) \end{aligned}$$

where

$$\begin{cases} \tilde{T} = \gamma(I - C\tilde{W}), \quad \tilde{W} = W(CW)^+ \\ \tilde{A} = A - \tilde{B}C \\ \tilde{A}_d = A_d - \tilde{B}_dC \\ \tilde{B} = BK + A\tilde{W}(I - DK) \\ \tilde{B}_d = A_d\tilde{W}(I - DK) \end{cases} \tag{21}$$

Next, conditions under which the above RUIO becomes a reduced-order observer are examined. Defining a reduced-order state transformation:

$$\bar{x}(t) = \Phi\tilde{x}(t) \tag{22}$$

where Φ is an arbitrary full-row rank matrix such that $dim(\bar{x}) < dim(\tilde{x})$. If there exists matrices C^*, A^*, and A_d^* such that the following conditions hold:

$$KC = KC^*\Phi, \quad \tilde{A} = A^*\Phi, \quad \tilde{A}_d = A_d^*\Phi \tag{23}$$

the RUIO (19)-(20) becomes the following reduced-order RUIO (RRUIO):

$$\begin{aligned} d(t) &= K(y(t) - C^*\bar{x}(t)) && (24) \\ \dot{\bar{x}}(t) &= \Phi A^*\bar{x}(t) + \Phi A_d^*\bar{x}(t-\tau(t)) + \Phi\tilde{B}y(t) \\ &\quad +\Phi\tilde{B}_dy(t-\tau(t)) && (25) \end{aligned}$$

Finally, it remains to solve the unknown matrices in (23). To achieve this, the first equation in (23) is reformulated as follows:

$$K(C\Phi^+ - C^*) = K\Theta = 0 \tag{26}$$

where matrix Θ is an arbitrary null space of K. Upon obtaining this Θ, we can obtain C^* from (26) as follows:

$$C^* = C\Phi^+ - \Theta \tag{27}$$

Then, the unknown matrices A^* and A_d^* are obtained from (23) as follows:

$$A^* = \tilde{A}\Phi^+, \quad A_d^* = \tilde{A}_d\Phi^+ \tag{28}$$

C. Comparison with The ROO's design

To establish the equivalence between the RRUIO and the ROO, a special case of the former is considered. Let the system matrices in (8) satisfy $B_1 = 0$, $B_2 = B$, $D_2 = D$ and $u_2 = u$. Thus, the corresponding RST (9) is given by:

$$\tilde{x}(t) = x(t) - Wf^*(t,x) \tag{29}$$

Using (29), the parameters of the RS (10) are modified as:

$$d(t) = u(t), \quad G = B, \quad G_d = 0, \quad H = D \tag{30}$$

To facilitate the following discussions, the derivation of the ROO is briefly reviewed. Let T be a full-row rank matrix such that $det\begin{bmatrix} TE \\ \bar{C} \end{bmatrix} \neq 0$ and $TW = 0$. Thus, the ROO state $\omega(t)$ can be asymptotically reconstructed by $TEz(t)$. Defining the following nonsingular transformation:

$$\begin{bmatrix} R \\ \bar{C} \end{bmatrix} = \begin{bmatrix} I & V \\ 0 & I \end{bmatrix}\begin{bmatrix} TE \\ \bar{C} \end{bmatrix} \tag{31}$$

where R is a full-row rank matrix and V is an arbitrary matrix, the following relationship holds:

$$TE + V\bar{C} = R \tag{32}$$

From (32) and the constraint $TW = 0$, the following relationship is obtained:

$$\begin{bmatrix} T & V \end{bmatrix} S = \begin{bmatrix} R & 0 \end{bmatrix} \tag{33}$$

where

$$S = \begin{bmatrix} E & W \\ \bar{C} & 0 \end{bmatrix} \tag{34}$$

By choosing a suitable value of R and solving (33) for T, general form of T can be obtained as:

$$T = \left(\begin{bmatrix} R & 0 \end{bmatrix} S^+ + Z(I - SS^+)\right)\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{35}$$

Here, Z is a matrix such that the following error vector:

$$\dot{\varepsilon}(t) = N\varepsilon(t) + N_d\varepsilon(t - \tau(t)) \tag{36}$$

converges asymptotically to zero, which is, however, achieved by solving a specific LMI problem (Theorem 4 in [12]). Once this Z matrix is obtained, the matrix T can be obtained from (35). The parameters of the ROO are then calculated via (5) using T:

$$\begin{bmatrix} N & L \\ N_d & L_d \end{bmatrix} = \begin{bmatrix} T\bar{A}M & T\bar{A}F \\ T\bar{A}_dM & T\bar{A}_dF \end{bmatrix} \tag{37}$$

$$M = \begin{bmatrix} R \\ \bar{C} \end{bmatrix}^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{38}$$

$$F = \begin{bmatrix} TE \\ \bar{C} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{39}$$

It is noted that ROO serves as an optimal observer for the SISE problem if and only if the rank condition (3) and the following one (Corollary 1 in [12]):

$$\begin{cases} rank\begin{bmatrix} sI_n - A - A_d & -B & W \\ C & D & 0 \end{bmatrix} = n + q + d \\ \qquad \forall s \in C, \quad Re(s) \geq 0 \end{cases} \tag{40}$$

are satisfied.

Finally, the equivalence of the RRUIO and the ROO can now be established by verifying: (1) the state of the ROO is a linear transformation of that of the RRUIO and (2) the input estimate obtained using the RRUIO is equivalent to that of the ROO under the restricted condition $rank[D] = q$. Using the condition $TW = 0$ and the following linear transformations:

$$\omega(t) = Tx(t), \quad x(t) = \tilde{x}(t) + Wf^*(t,x) \tag{41}$$

the relationship between the states of the ROO and the RRUIO is given by:

$$\omega(t) = T\tilde{x}(t) = \bar{T}\bar{x}(t) \tag{42}$$

where $T = \bar{T}\Phi$, in which $\bar{T}$ is nonsingular. Using (42), the state equation of the ROO (4) can be directly derived from that of the RRUIO:

$$\begin{aligned} \dot{\omega}(t) &= \bar{T}\dot{\bar{x}}(t) \\ &= \bar{T}\Phi A^*\bar{x}(t) + \bar{T}\Phi A_d^*\bar{x}(t - \tau(t)) + \bar{T}\Phi\tilde{B}y(t) \\ &\quad + \bar{T}\Phi\tilde{B}_d y(t - \tau(t)) \\ &= \underbrace{\bar{T}\Phi A^*\bar{T}^{-1}}_{N}\omega(t) + \underbrace{\bar{T}\Phi A_d^*\bar{T}^{-1}}_{N_d}\omega(t - \tau(t)) \\ &\quad + \underbrace{\bar{T}\Phi\tilde{B}}_{L}y(t) + \underbrace{\bar{T}\Phi\tilde{B}_d}_{L_d}y(t - \tau(t)) \end{aligned} \tag{43}$$

The input estimates of the ROO can be obtained from those of the RRUIO using (24):

$$\begin{aligned} u(t) &= -K\bar{C}\bar{x}(t) + Ky(t) \\ &= \underbrace{-K\bar{C}\bar{T}^{-1}}_{\begin{bmatrix} 0 & I \end{bmatrix} M}\omega(t) + \underbrace{K}_{\begin{bmatrix} 0 & I \end{bmatrix} F}y(t) \end{aligned} \tag{44}$$

Note that the above derivation of the ROO provides an alternative to the original approach in (31)-(39), which requires selecting a full-row rank matrix R and solving an LMI to determine a full-row rank matrix T. On the other hand, the proposed RRUIO design relies only on standard unknown input filtering techniques.

D. Finite Precision Consideration

It is noted that the input estimate obtained using the proposed RRUIO is more compact than that of the ROO from the perspective of finite-precision implementation. This advantage arises because some system parameters in the ROO are determined using the inverse matrix of $\bar{T}$, which may involve a large number of floating-point operations. Such operations can increase the difficulty of hardware implementation [13], e.g., when implementing an image encryption algorithm on an FPGA platform (see Section IV.B for an illustration).

IV. AN ILLUSTRATIVE EXAMPLE

A. Equivalence of the RRUIO and the ROO

Consider the numerical example presented in [12], where the system matrices are given as follows:

$$A = \begin{bmatrix} -10 & 1 & 2 \\ -48 & -2 & 0 \\ 1 & -1 & -20 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.5 & 0 & -1 \\ -0.5 & 1 & 0.5 \\ 0.25 & 0 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \ \tau(t) = 0.4s, \ \alpha = 0, \ f(t,x) = tx_2(t)x_3(t)$$

The input $u(t)$ is an unknown input signal defined as $u(t) = sin(2t)$ in the simulation. The simulation time is set to $20s$.

The ROO is first designed by selecting:

$$R = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0.5 & -1 & 0 & 1 \end{bmatrix} \tag{45}$$

Using (38), matrix M is obtained as:

$$M = \begin{bmatrix} 0.1379 & 0.0690 \\ -0.2759 & -0.1379 \\ 0.3103 & -0.3448 \\ -0.3448 & 0.8276 \end{bmatrix} \tag{46}$$

Then, using (35), matrix T is computed as:

$$T = \begin{bmatrix} 5 & 0 & 1 \\ 4.5 & 0 & -2 \end{bmatrix} \tag{47}$$

where $Z = 0$ is chosen since $I - SS^+ = 0$. Finally, using (37), (39), (46), and (47), the resulting ROO is obtained as:

$$\begin{cases} \dot{\omega}(t) = \underbrace{\begin{bmatrix} -13.3793 & 5.3103 \\ 6.7586 & -20.6207 \end{bmatrix}}_{N} \omega(t) \\ \quad + \underbrace{\begin{bmatrix} -1.0172 & 1.7414 \\ -1.4655 & 2.0172 \end{bmatrix}}_{N_d} \omega(t-0.4) \\ \quad + \underbrace{\begin{bmatrix} -6 & 10 \\ 12 & -5.5 \end{bmatrix}}_{L} y(t) \\ u(t) = \underbrace{\begin{bmatrix} -0.345 & 0.828 \end{bmatrix}}_{\begin{bmatrix} 0 & I \end{bmatrix} M} \omega(t) + \underbrace{\begin{bmatrix} -2 & 2 \end{bmatrix}}_{\begin{bmatrix} 0 & I \end{bmatrix} F} y(t) \end{cases} \tag{48}$$

Next, the design of the RRUIO is illustrated as below. First, the RST in (29) is applied, by which the RS in (10) is obtained as follows:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + A_d\tilde{x}(t-0.4) + Gd(t) \\ \quad + \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{AW} f^*(t,x) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{A_dW} f^*(t-0.4,x) \\ y(t) = C\tilde{x}(t) + Hd(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{CW} f^*(t,x) \end{cases} \tag{49}$$

where $G = B$, $G_d = 0$, $d = u$, and $H = D$. Next, using (11), (14), (15), and (49), we can obtain the disturbance and unknown input vectors, respectively, as follows:

$$f^*(t,x) = \underbrace{\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}}_{(CW)^+} (y(t) - C\tilde{x}(t) - Hd(t)) \tag{50}$$

$$d(t) = \underbrace{\begin{bmatrix} -2 & 2 \end{bmatrix}}_{(\tilde{T}H)^+\tilde{T}} (y(t) - C\tilde{x}(t)) \tag{51}$$

where $\tilde{T} = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}$, in which the choosing matrix γ in (15) is set by $\gamma = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Using (19)-(21), the RUIO is obtained as:

$$\begin{cases} \dot{\tilde{x}}(t) = \underbrace{\begin{bmatrix} -10 & 0 & 0 \\ -44 & 0 & 0 \\ 7 & 0 & -24 \end{bmatrix}}_{\tilde{A}} \tilde{x}(t) \\ \quad + \underbrace{\begin{bmatrix} 0.5 & 0 & -1 \\ -2.5 & 0 & 0.5 \\ 0.25 & 0 & 0.5 \end{bmatrix}}_{\tilde{A}_d} \tilde{x}(t-0.4) + \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & 2 \\ -6 & 5 \end{bmatrix}}_{\tilde{B}} y(t) \\ \quad + \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}}_{\tilde{B}_d} y(t-0.4) \\ d(t) = \underbrace{\begin{bmatrix} 2 & 0 & -2 \end{bmatrix}}_{-KC} \tilde{x}(t) + \underbrace{\begin{bmatrix} -2 & 2 \end{bmatrix}}_{K} y(t) \end{cases} \tag{52}$$

Then, choosing the reduced-order state transformation matrix Φ arbitrary and solving (26) for Θ, we can obtain:

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Theta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{53}$$

From (27) and (28), we can determine C^*, A^*, and A_d^* as follows:

$$C^* = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad A^* = \begin{bmatrix} -10 & 0 \\ -44 & 0 \\ 7 & -24 \end{bmatrix},$$

$$A_d^* = \begin{bmatrix} 0.5 & -1 \\ -2.5 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

Accordingly, the RUIO in (52) can be implemented more efficiently using (24)-(25) as the following RRUIO:

$$\begin{cases} u(t) = \underbrace{\begin{bmatrix} 2 & -2 \end{bmatrix}}_{-KC^*} \bar{x}(t) + \underbrace{\begin{bmatrix} -2 & 2 \end{bmatrix}}_{K} y(t) \\ \dot{\bar{x}}(t) = \underbrace{\begin{bmatrix} -10 & 0 \\ 7 & -24 \end{bmatrix}}_{\Phi A^*} \bar{x}(t) \\ \quad + \underbrace{\begin{bmatrix} 0.5 & -1 \\ 0.25 & 0.5 \end{bmatrix}}_{\Phi A_d^*} \bar{x}(t-0.4) + \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}}_{\Phi\tilde{B}} y(t) \end{cases} \tag{54}$$

Finally, to verify the equivalence between (48) and (54), the linear transformation relating the ROO and the RRUIO states is given by:

$$\omega(t) = \underbrace{\begin{bmatrix} 5 & 1 \\ 4.5 & -2 \end{bmatrix}}_{\bar{T}} \bar{x}(t) \tag{55}$$

through which, the equivalence of (48) and (54) can be easily verified.

B. Input Estimation Within a Finite Precision Environment

As shown in the previous subsection, the ROO and the RRUIO exhibit identical filtering performance when no explicit restrictions are imposed on the system parameters of the ROO. However, this equivalence holds only under ideal numerical conditions. In some floating-point restricted computing structures, such as fixed-point arithmetic implementations on FPGA platforms, the two observers may exhibit different behaviors. To simplify the discussions, it is assumed that all system parameters are limited to one decimal place. Under this assumption, the RRUIO remains unchanged, whereas the implementation of the ROO is modified as follows:

$$\begin{cases} \dot{\omega}(t) = \begin{bmatrix} -13.38 & 5.31 \\ 6.76 & -20.62 \end{bmatrix} \omega(t) \\ \quad + \begin{bmatrix} -1.02 & 1.74 \\ -1.47 & 2.02 \end{bmatrix} \omega(t-0.4) \\ \quad + \begin{bmatrix} -6 & 10 \\ 12 & -5.5 \end{bmatrix} y(t) \\ u(t) = \begin{bmatrix} -0.34 & 0.83 \end{bmatrix} \omega(t) + \begin{bmatrix} -2 & 2 \end{bmatrix} y(t) \end{cases} \tag{56}$$

The unknown input filtering performances of (48) and (54) are then obtained, respectively, as follows:

$$RMSE_u(ROO) = 0.0149, \quad RMSE_u(RRUIO) = 0.0017$$

These results indicate that the filtering performance of the ROO is slightly worse compared to that of the RRUIO under finite-precision constraints.

In summary, the proposed RRUIO has more compact system matrices than those of the ROO.

V. CONCLUSIONS

This paper presents a refined unknown input observer (RUIO) design for time-delay systems with uncertainties, with the objective of enhancing encryption performance in secure communication applications. The ASISE approach for linear time-invariant time-delay uncertain systems with unknown inputs is employed to facilitate the design of an unknown input observer. For precision-limited computing environments, the proposed reduced-order RUIO (RRUIO) yields more compact system matrices than those of a conventional design. Simulation RMSE results confirmed the effectiveness of the proposed approach.

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