Stabilization and Control of Dynamic Systems in Metzlerian Block Controllable Canonical Form

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***Abstract*— This paper introduces a special class of dynamic systems represented by block controllable canonical form, whereby its blocks are strictly Metzler matrices. Such systems appear in diverse applications including electromechanical systems, circuits, structural vibration control, robotics, and aerodynamics, in which stabilization and robust performance are required. Since the coefficient matrices in their model representations have special structures, recognized as Metzler matrices, it is important to maintain their structural properties while designing the control systems. We initially provide an important stability result pertaining to continuous-time positive systems known as Metzlerian systems. Then, we define dynamic systems in Metzlerian block controllable canonical form (MBCCF) and investigate their stability. Finally, we outline procedures for stabilization of systems in block controllable canonical form using state feedback control law such that the closed-loop systems are imposed to be in MBCCF.**

I. Introduction

The models of many dynamical systems are described by second- or higher-order vector differential equations. Their state-space representations are commonly written in block controllable canonical form (BCCF). Block modal matrices and BCCF tie with polynomial matrix theory and matrix fraction description. They provide important insights on stability analysis and state space realization of multi-variable dynamic systems [1]–[4]. As pointed out in the abstract of this paper, a collection of various applications admit these types of model structures, which require robust stability and control design [5]–[12]. It is interesting to note that their associated system matrices are in block companion form with blocks of Metzler matrices, which may also be symmetric. Metzler matrices have very impressive stability properties, and they are associated with continuous-time positive systems or Metzlerian systems. Positive systems are important class of dynamic systems that have their own special appearance in many diverse applications [13], [14]. The response of positive systems to positive excitations remains positive. Stability robustness, positive stabilization, and control design of positive systems have been reported in [15]–[23], which include extensive references. This paper attempts to establish a generalization of Metzler systems to BCCF and to provide stabilization and eigenvalue assignment techniques for them.

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We initially extend the previous results on stability of Metzler matrices to the polynomial matrices with coefficients belonging to the class of M-matrices. After summarizing preliminary results in Section II, we give a new result on stability of Metzlerian block controllable canonical form in section III. Section IV outlines the available results for Metzlerian stabilization of a regular system using state feedback. We take advantage of it as an initial step to solve the unsolved problem of eigenvalue assignment for Metzlerian systems. Two possible procedures are obtained, which require several transformations and conversions to allow qualified eigenvalues to be assigned. Finally, Section V provides stabilization and eigenvalue assignment for systems in BCCF and MBCCF. Numerical examples in Section VI support the theoretical results. Before closing this section, we define the notation > 0 (≥ 0) for positive (nonnegative) numbers, vectors, and matrices. Thus, $A > 0$ ($A \geq 0$) represents a positive (nonnegative) matrix, which is distinguished from $A \succ 0$ ($A \succeq 0$) for a positive definite (positive semidefinite) symmetric matrix. Furthermore, we define the notation $|A|$ for det in section III and $|A(i,j)|$ for elementwise magnitude of a matrix.

II. Preliminary Development and Basic Results

Many dynamical systems in applications are modeled by high–order vector–valued differential equations of the form

$$y^{(r)}(t) + \sum_{i=0}^{r-1} A_i\, y^{(i)}(t) = \sum_{i=0}^{l} B_i\, u^{(i)}(t), \qquad (l \leq r) \quad (1)$$

where $y \in \mathbb{R}^m$ and $u \in \mathbb{R}^q$ are output and input vectors, respectively. The associated coefficient matrices are $A_i \in \mathbb{R}^{m\times m}$ and $B_i \in \mathbb{R}^{m\times q}$. For simplicity, we assume $q = m$ and $l = r$. If we denote $A(s) = I_m s^r + \sum_{i=0}^{r-1} A_i s^i$, and $B(s) = \sum_{i=0}^{l} B_i s^i$, then (1) can be written as $A(s)y(s) = B(s)u(s)$, and the transfer function from $u(s)$ to $y(s)$ is given by a matrix fraction description (MFD), $G(s) = A(s)^{-1}B(s)$.

The characteristic equation of the open–loop system is $\alpha(s) = \det A(s)$ with $\deg(\alpha(s)) = mr$. Assuming the system (1) is unstable, one immediate problem of interest is to stabilize the system by pole assignment using state feedback controller of the form

$$u(t) = f\{\, y(t), \ldots, y^{(r-1)}(t);\, u(t), \ldots, u^{(l-1)}(t)\,\} \quad (2)$$

such that the closed–loop system consisting of (1) and (2) admits a set of desired poles. This problem can conveniently be reformulated in terms of state–space realizations of (1).

The solution of the problem require both the derivative of inputs and outputs in state feedback control law (2), which has been treated in [24], [25]. Now, let us consider the special case of $l = 0$ and obtain the following first–order state–space model of high–order vector–valued differential equation

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{r-1} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0_m \\ 0_m \\ \vdots \\ 0_m \\ B_0 \end{bmatrix}}_{B} u(t) \tag{3}$$

where $y(t) = Cx(t)$ with

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(l-1)}(t) \end{bmatrix} \in \mathbb{R}^n, \quad n = r \times m.$$

The state–space representation (3) is known as the *block controllable canonical form*. In this paper, we concentrate on a special structure of (3) with $-A_i$'s, $(i = 0, 1, \ldots, r-1)$ constrained to be strictly Metzler matrices.

There exist a collection of electromechanical systems that are modeled by a special structure of the form (3) with blocks of strictly Metzler matrices. For example, a model of a drive train containing three flywheels, two dash pots and two springs can be represented by $J\ddot{\theta} + D\dot{\theta} + S\theta = \tau$, where θ is a vector of angular displacement, τ is the torque vector, J is a diagonal inertia matrix, and the pair $\{D, S\}$ represent symmetric damping and stiffness matrices, given by

$$D = \begin{bmatrix} d_1 & -d_1 & 0 \\ -d_1 & d_1 + d_2 & -d_2 \\ 0 & -d_2 & d_2 \end{bmatrix}, S = \begin{bmatrix} s_1 & -s_1 & 0 \\ -s_1 & s_1 + s_2 & -s_2 \\ 0 & -s_2 & s_2 \end{bmatrix}$$

The same type of matrices appear in mechanical systems described by $M\ddot{y} + D\dot{y} + Sy = u$, where M, D, S are symmetric matrices of mass, dash pot, and spring; u is the forcing vector function; and y is the displacement vector.

The matrices with off–diagonal nonnegative and diagonal positive entries are known as M–matrices. The negatives of such matrices become strictly Metzler matrices, which we will formally define them below and elaborate on their connections to the special structure of dynamic systems represented by (3).

Definition 1. A matrix is called Metzler if its off-diagonal elements are nonnegative i.e. a_{ij}, $i \neq j$. It is called strictly Metzler if in addition $a_{ij} < 0$, which satisfies the necessary condition of Hurwitz stability of A.

Definition 2. A dynamic system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ with a Metzler matrix A and nonnegative matrices $B \geq 0$, $C \geq 0$, $D \geq 0$ is called positive continuous-time system or Metzlerian system. The response of Metzlerian system to positive initial condition and positive inputs remains in positive orthant of state space.

Theorem 1. *The Metzlerian system* (4) *is asymptotically stable if and only if the following equivalent conditions is satisfied:*

1) *The matrix A is strictly diagonal dominant i.e. $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$, $i \neq j$.*
2) *All leading principal minors of $-A$ are positve.*
3) *The matrix A is nonsingular and $-A^{-1} > 0$.*
4) *There exists a positive definite diagonal matrix P such that $A^T P + PA \prec 0$.*
5) *There exists a positive vector v such that $Av < 0$.*

Definition 3. The system (3) with strictly Metzler matrices $-A_i$'s is formally defined as the *Metzlerian block controllable canonical form (MBCCF).*

It is important to note that the Hurwitz stability of strictly Metzler matrices $-A_i$'s in (3) does not imply Hurwitz stability of the matrix A. Therefore, it is of interest to establish simple stability conditions in terms of A_i's. Let us consider the scalar–valued differential equation from (1) by replacing $\{A_i, B_i\}$ with $\{a_i, b_i\}$ and let $r = n$ with proper assumption $l = n$. Then, defining $a(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$, $b(s) = \sum_{i=0}^{n} b_i s^i$, the transfer function $G(s) = \frac{y(s)}{u(s)} = \frac{b(s)}{a(s)}$ admits a state–space realization in the standard controllable canonical form (CCF), where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{5}$$

$$C = \begin{bmatrix} b_0 - a_0 b_n, & \cdots & , b_{n-1} - a_{n-1} b_n \end{bmatrix}, \quad D = b_n$$

We may refer to (5) as Metzlerian controllable canonical form, which is the special case of (3) with $-a_i$'s < 0 representing scalar Metzler terms. This implies the necessary stability condition of (5). Thus, the Metzlerian block companion matrix A in (3) is reduced to the Metzler companion matrix in (5).

III. Stability of Metzlerian Block Controllable Canonical Form (MBCCF)

The stability of MBCCF can be investigated using the polynomial matrix

$$A(s) = I_m s^r + \sum_{i=0}^{r-1} A_i s^i \tag{6}$$

The stability of $A(s)$ is determined by the application of Hurwitz criterion to the polynomial

$$\alpha(s) = \det(A(s)) = s^n + \sum_{i=0}^{n-1} \alpha_i s^i, \quad n = \deg(d(s)) = mr \tag{7}$$

However, it is of interest to obtain simple stability conditions in terms of A_is. Initial effort shows the possibility of achieving this goal, which requires careful analysis. For the important second order polynomial matrix, simplified stability conditions can be established.

Lemma 1. *The polynomial matrix $A(s) = A_2 s^2 + A_1 s + A_0$ associated with the second order vector differential equation representation of a dynamic system is stable if symmetric matrices A_i's, $i = 0, 1, 2$, are positive definite. Alternatively, if $-A_i$'s are stable Metzler matrices, then the system is stable. Furthermore, the system is positively stable if in addition $|A_1(i,j)| > |A_0(i,j)|$ with the assumption of $A_2 = I_m$ based on the defined notation of* (3) *and* (6).

The above lemma shows that the response of positively stable second-order MBCCF to positive initial condition and positive input remains in the nonnegative orthant of the state space. Thus, one can conclude the possibility of connecting systems with companion system matrix to Metzlerian systems. See example 1 for more details.

IV. Stabilization of Metzlerian Systems

A. Stabilization of Metzlerian Systems by State Feedback

The problem of stabilization of Metzlerian systems by state feedback has been solved completely based on LMI and LP in [17]–[19]. We recap the main result for the purpose of its utilization in subsequent sections.

Let the state feedback control law

$$u(t) = v(t) + Kx(t) \tag{8}$$

be applied to the system (4). Then the closed-loop system becomes

$$\dot{x} = (A + BK)x + Bv \tag{9}$$

Theorem 2. *There exists a state feedback control law* (8) *for the system* (4) *such that the closed-loop system* (9) *becomes Metzlerian stable if and only if*

1) *The following LMI has a feasible solution with respect to the variables Y and Z:*

$$AZ + BY + ZA^T + Y^T B^T \prec 0 \tag{10}$$

$$(AZ + BY)_{ij} \geq 0 \quad \text{for } i \neq j \tag{11}$$

where $Z \succ 0$ is a diagonal positive definite matrix. Furthermore, the gain matrix K is obtained from

$$K = YZ^{-1} \tag{12}$$

2) *Or, the following LP has a feasible solution with respect to the variables $z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^m$ for all $i = 1, \ldots, n$:*

$$Az + B\sum_{i=1}^{n} y_i < 0, \quad z > 0 \tag{13}$$

$$a_{ij} z_j + b_i y_j \geq 0, \quad i \neq j \tag{14}$$

where $A = [a_{ij}]$ and $B = [b_1, b_2, \ldots, b_n]^T$. Furthermore, the gain matrix K is obtained from

$$K = \left[\frac{y_1}{z_1}, \frac{y_2}{z_2}, \ldots, \frac{y_n}{z_n}\right] \tag{15}$$

B. Eigenvalue Assignment of Metzlerian Systems

The problem of eigenvalue assignment for Metzlerian systems is not trivial and no concrete result is available to solve the problem. The best known result was reported for single-input discrete-time positive systems represented in controllable canonical form [26]. Using [26], we propose an indirect design procedure for eigenvalue assignment of continuous-time positive systems (Metzlerian systems).

Indirect Design Procedure:

1) Let the single-input unstable Metzlerian system (4) be stabilized using Theorem 1, and define the stabilized system by the pair $\{A_s, B_s\}$ and specify the feasible eigenvalue region. This initial step guarantees that positive stabilization is possible.
2) Convert the stable continuous-time system pair $\{A_s, B_s\}$ to discrete-time pair $\{A_d, B_d\}$ using the standard state space conversion given by $A_d = e^{A_s T}$, $B_d = \int_0^T e^{\alpha A_s} B_s d\alpha = A_s^{-1}(e^{A_s T} - I)B_s$. Note that $A_d \geq 0$ is a nonnegative matrix when A_s is Metzler. Similarly, $B_d \geq 0$ using the above expression. Also, convert λ_i's to discrete eigenvalues μ_i's.
3) Select the desired eigenvalue λ_i's and convert them to discrete eigenvalues μ_i's using bilinear transformation.
4) Transform the pair $\{A_d, B_d\}$ to CCF and apply the eigenvalue assignment technique of [26] provided the associated conditions on μ_i's are satisfied to obtain K_d.
5) Convert $A_d + B_d K_d$ back to continuous-time closed-loop system matrix $A_c = A_s + B_s K$ and recover K.

Remark 1. Successful completion of the above procedure up to Step 4 guarantees that the desired selection of eigenvalues can be employed after step 1 for the pair $\{A_s, B_s\}$. Thus, one may use the entire eigenstructure assignment to recover K by

$$K = [q_1 \quad q_2 \quad \ldots \quad q_n][v_1 \quad v_2 \quad \ldots \quad v_n]^{-1} \tag{16}$$

where v_i and q_i are obtained from the kernel or null space of the matrix equation

$$\begin{bmatrix} A_s - \lambda_i I & B_s \end{bmatrix} \begin{bmatrix} v_i \\ q_i \end{bmatrix} = 0, \quad i = 1, 2, \ldots, n \tag{17}$$

which is directly derived from closed-loop eigenvalues and eigenvector relationship $(A + BK)v_i = \lambda_i v_i$, with $q_i = Kv_i$.

V. Stabilization of MBCCF Systems by Eigenvalue Assignment

A. Constrained Stabilization of Systems in BCCF

The systems in BCCF are constrained by their structures as described in Section II, whereby block matrices A_i's are M-matrices, which make $-A_i$'s Metzler matrices. Due to the appearance of physical parameters such as stiffness and damping terms in these block matrices, it is required to maintain their structures when modifying the associated parameters in control design. This observation leads to the fact that the control law

$$u = v + Kx = v + [K_0 \quad K_1 \quad \ldots \quad K_{r-1}]x \tag{18}$$

be applied to the controllable system represented by BCCF (3). Assuming for simplicity $B_0 = I_m$, the closed-loop system matrix denoted by $\hat{A} = A + BK$ should be stable and preserve the block companion structure, where

$$-\hat{A}_i = -A_i + K_i, \quad \text{for } i = 0, 1, \ldots, r-1 \tag{19}$$

We propose the following lemma to generate the stable matrix $\hat{A}$.

Lemma 2. *Let us define a system with the pair $\{\tilde{A}, \tilde{B}\}$ extracted from the BCCF* (3)*, where*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 \\ 0 \end{bmatrix}, \quad \tilde{B} = B = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \tag{20}$$

with $\tilde{A}_1$ representing the first $n - m$ rows of the block companion matrix A in (3)*. Then there exists a matrix $\tilde{A}_2$ such that*

$$\hat{A} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \tag{21}$$

is Hurwitz stable. Furthermore, the required feedback gain matrix can be obtained by $\hat{A} = A + BK$ with the aid of (19).

Proof: The proof is obvious by writing

$$\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{A}_1 \\ 0 \end{bmatrix}}_{\bar{A}} + \underbrace{\begin{bmatrix} 0 \\ I_m \end{bmatrix}}_{\bar{B}} \tilde{A}_2 \tag{22}$$

Then, there exists a matrix $\tilde{A}_2$ such that (22) is Hurwitz stable by employing an eigenvalue assignment to the pair $\{\bar{A}, \bar{B}\}$. The resulting matrix $\tilde{A}_2$ specifies the blocks $-\hat{A}_i$ of the matrix $\hat{A}$. Consequently, K can be obtained from (19).

Lemma 2 is a simple elegant result that can be used for stabilization of systems represented by BCCF (3). However, the distribution of blocks $\hat{A}_i$ in the matrix $\hat{A}$ does not impose any further constraints on their structures, e.g., $-\hat{A}_i$'s to be Metzler matrices. The next section provides a method to achieve MBCCF stabilization by eigenvalue assignment.

To achieve constrained stabilization of systems in BCCF with stable block Metzler matrices, one can construct stable Metzler matrices with a given set of desired eigenvalues. The stability result of Section III facilitates to construct stable block companion matrices $\hat{A}$ with guaranteed stable block Metzler matrices. This can be done by a sequence of stabilization process using section IV applied to A_i's. However, this approach is rather lengthy for high-order systems. It is convenient to be used for second-order systems in BCCF with the aid of lemma 1. The following section is a more elegant way to perform stabilization with eigenvalue assignment for systems in BCCF or MBCCF.

B. Stabilization by Eigenvalue Assignment for Systems in MBCCF

It is evident that if the state feedback control law (18) is applied to the system (3) represented by BCCF, it should preserve the BCCF structure of the closed-loop system. Therefore,

$$\hat{A} = A + BK$$

is required to be a stable Metzlerian block companion matrix, whereby $-\hat{A}_i$'s are strictly Metzler matrices.

Let the unstable system (3) with the pair $\{A, B\}$ is either in BCCF or MBCCF. For simplicity we assume that $B_0 = I_m$, otherwise a simple transformation is needed to realize it. Clearly, the corresponding characteristic polynomial matrix of $\hat{A}$ is

$$\hat{\Delta}(\lambda) = I_m \lambda^r + \sum_{i=0}^{r-1} \hat{A}_i \lambda^i \tag{23}$$

We define the block structure matrix

$$F = \begin{bmatrix} F_0 & I_m & 0 & \cdots & 0 \\ 0 & F_1 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \\ 0 & 0 & \cdots & 0 & F_{r-1} \end{bmatrix} \tag{24}$$

with stable block diagonal matrices F_i's. The matrix, F is considered as a linearization of $\hat{\Delta}(\lambda)$ and the coefficients of $\hat{\Delta}(\lambda)$ can be related to F_i's using the following theorem.

Theorem 3. *The stable Metzlerian block companion matrix $\hat{A}$ defined by $\hat{A}_i$'s is similar to the matrix F, i.e. $F = P\hat{A}P^{-1}$, where the transformation matrix P is a lower triangular matrix with (i, j) blocks $P_{ij} = I_m$ for $i \neq j$ and P_{ij} for $i > j$ satisfying a set of chain equations.*

Proof: One can establish the proof by substituting the defined matrices into $FP = P\hat{A}$. Here, we show this for one special case with

$$\hat{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\hat{A}_0 & -\hat{A}_1 & -\hat{A}_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_0 & I & 0 \\ 0 & F_1 & I \\ 0 & 0 & F_2 \end{bmatrix},$$
$$P = \begin{bmatrix} I & 0 & 0 \\ P_{21} & I & 0 \\ P_{31} & P_{32} & I \end{bmatrix} \tag{25}$$

Thus, we substitute $\hat{A}, F, P$ matrices in $FP = P\hat{A}$ and obtain the following set of chain equations,

$$\begin{aligned} & F_0 + P_{21} = 0, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & F_1 P_{21} + P_{31} = 0, \\ & P_{21} - F_1 - P_{32} = 0, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \hat{A}_0 = -F_2 P_{31}, \\ & \hat{A}_1 = P_{31} - F_2 P_{32}, \\ & \hat{A}_2 = P_{32} - F_2 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{26}$$

The chain equation can solve for the unknown P_{ij} and subsequently $\hat{A}_i$'s can be computed. This theorem shows that by defining the desired set of eigenvalues and distributing them in matrix F, one can easily obtain a stable block companion matrix with the same desired eigenvalues. Consequently, the following lemma is an immediate conclusion of Theorem 3.

Lemma 3. *Let F_i's, $i = 0, \ldots, r-1$ be chosen such that $\bigcup_{i=0}^{r-1} \lambda(F_i) = \Omega$, where Ω is the prescribed set of desired eigenvalues. Then the $m \times m$ coefficient matrices of* (23) *are given by*

$$-\hat{A}_{r-j} = (-1)^{j+1} \hat{S}_j(F_0, F_1, \ldots, F_{r-1}) \tag{27}$$

where $\hat{S}_j$ can be considered as a generalized elementary symmetric function associated with the set F_i's.

The proof of this lemma is obvious by equating $\hat{\Delta}(\lambda)$ in (23) to $\prod_{i=0}^{r-1}(\lambda I - F_i)$. Let us apply this for the special case of (25). So, we have

$$\begin{aligned}\hat{\Delta}(\lambda) &= (\lambda I - F_0)(\lambda I - F_1)(\lambda I - F_2) \\ &= I\lambda^3 - \underbrace{(F_0 + F_1 + F_2)}_{\hat{A}_2} \lambda^2 \\ &\quad + \underbrace{(F_0F_1 + F_0F_2 + F_1F_2)}_{\hat{A}_1} \lambda - \underbrace{F_0F_1F_2}_{\hat{A}_0},\end{aligned} \tag{28}$$

and (27) follows for the general case.

Using the above development, we summarize the following design steps for eigenvalue assignment of systems in MBCCF.

1) Choose the set of desired eigenvalues Ω.
2) Construct stable Metzler matrices F_i's such that $\bigcup_{i=0}^{r-1} \lambda(F_i) = \Omega$, Select F_i's to guarantee Metzler structures for $-A_i$ (see below for selection strategy).
3) Compute $-A_i$'s using (27).
4) Obtain the state feedback gain matrix K from (19).

The strategy for selecting F_i's in Step 2 should be a set of block diagonal stable matrices, each with multiple Metzler blocks of order 2 or 1 such that, for even or odd m, the blocks are completely distributed. It is easy to show that for $m = 2$, the product of two Metzler matrices becomes an M-matrix; the product of an M-matrix with a Metzler matrix remains a Metzler matrix; and the sum of two or more Metzler matrices (alternatively, M-matrices) maintain its structure. Since (27) is constructed based on sum and product of Metzler matrices, $-A_i$'s remain Metzlerian. Note that for $m > 2$, define block diagonal stable Metzler matrices F_i's such that each block is fit with Metzler matrices of size 2 or 1.

VI. Illustrative Examples

Example 1: Let (3) be in MBCCF with $r = 2$ and $m = 2$, where

$$A_0 = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}, \quad B_0 = I_2.$$

The stability of the system can be determined by the second-order matrix polynomial $A(s) = I_2 s^2 + A_1 s + A_0$, since A_0 and A_1 are positive definite Metzler matrices. Furthermore, using Lemma 1 one can also conclude that MBCCF representation of (3) is positively stable, since $|A_1(i,j)| > |A_0(i,j)|$ for all i, j. In fact, it is easy to plot the response of the system with positive initial condition and positive input and to confirm the positive response.

This example motivates one to consider the possibility of expanding the class of positive systems beyond Definition 2. Consider two single-input single-output systems with

$$\tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Both systems $\{\tilde{A}_1, B, C\}$ and $\{\tilde{A}_2, B, C\}$ are stable. However, the system $\{\tilde{A}_1, B, C\}$ admits positive response, since $a_1 = 4 > a_2 = 3$. If we transform $\{\tilde{A}_1, B, C\}$ with

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix},$$

the resulting system becomes Metzlerian with

$$\tilde{A}_1 = PAP^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad \bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\bar{C} = CP^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

So, all stable second-order systems in CCF with $a_1 > a_0$ represent positive systems.

Example 2: Consider the unstable BCCF

$$A = \begin{bmatrix} 0 & I_2 \\ -A_0 & -A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_2 \end{bmatrix},$$

where

$$-A_0 = \begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix}, \quad -A_1 = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}.$$

Let $\tilde{A}_1 = \begin{bmatrix} 0 & I_2 \end{bmatrix}$ and define (20). Then construct $\bar{A}$ from (21) with unknown $\tilde{A}_2$. Solving the stabilization problem (22) for the pair

$$\bar{A} = \begin{bmatrix} \tilde{A}_1 \\ 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}, \quad \tilde{A}_2 = \bar{K},$$

one possible solution was obtained as

$$\bar{K} = \begin{bmatrix} 2 & -1 & -7 & 1 \\ 3 & -1 & 1 & -1 \end{bmatrix},$$

which makes $\bar{A} + \bar{B}\bar{K}$ stable with eigenvalues -7.374, -0.4414, $-0.0920 \pm j0.5466$. The required state feedback gain matrix for the pair $\{A, B\}$ can be computed from (19), whereby $\bar{K} = \tilde{A}_2 = \begin{bmatrix} -\hat{A}_0 & -\hat{A}_1 \end{bmatrix}$ specifies the blocks $\hat{A}_1$ and $\hat{A}_2$. Thus,

$$K_0 = -\hat{A}_0 + A_0 = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix}.$$

$$K_1 = -\hat{A}_1 + A_1 = \begin{bmatrix} -7 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -9 & -3 \\ -2 & -8 \end{bmatrix},$$

and $K = [K_0 \ K_1]$ stabilizes $\hat{A} = A + BK$ and maintains the BCCF structure. Note that $\hat{A}$ is not in MBCCF. However, one can apply the design of section V-B to guarantee the structure of MBCCF. Since this system is in second order BCCF, one can also employ lemma 1 and apply twice stabilization of section IV for A_0 and A_1 to achieve MBCCF.

Example 3: Consider the following system in BCCF (3) with $m = 2$ and $r = 3$, where

$$A_0 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}.$$

It is not difficult to see that the system is unstable. Let us apply the procedure of Section V-B to stabilize the system by eigenvalue assignment such that the closed-loop system becomes MBCCF with desired eigenvalues -1, -2, -3, -4, -5, -6. Using these eigenvalues, the matrix F in (24) can be constructed with the following Metzler matrices:

$$F_0 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -5 & 1 \\ 0 & -6 \end{bmatrix}.$$

Applying Theorem 2 with the aid of (25) and (26), we get $P_{21} = -F_0, \quad P_{31} = F_1F_0, \quad P_{32} = -F_0 - F_1$, and

$$\hat{A}_0 = -(F_0 + F_1 + F_2) = \begin{bmatrix} 9 & -2 \\ -2 & 12 \end{bmatrix},$$

$$\hat{A}_1 = F_0F_1 + F_0F_2 + F_1F_2 = \begin{bmatrix} 25 & -14 \\ -14 & 46 \end{bmatrix},$$

$$\hat{A}_2 = -F_0F_1F_2 = \begin{bmatrix} 25 & -29 \\ -20 & 52 \end{bmatrix}.$$

Thus, the matrices $-\hat{A}_i$'s, $i = 1, 2, 3$ become strictly Metzler matrices and $\hat{A}$ will be the Metzlerian block companion matrix. Finally, the required state feedback gain matrix can be obtained from (19) as

$$K = \begin{bmatrix} -8 & 1 & -23 & 18 & -34 & 27 \\ 2 & -10 & 17 & -47 & 22 & -64 \end{bmatrix}.$$

Hence, the closed-loop system results in MBCCF with the desired eigenvalues.

VII. Conclusions

In summary, we introduced generalized Metzlerian systems formulated in block controllable canonical form and defined it as Metzlerian Block Controllable Canonical Form (MBCCF). A central stability criterion for continuous-time positive Metzlerian systems was used to serve as the foundation for the development of stabilization methods. A state-feedback control design strategy was outlined to demonstrate how the closed-loop dynamics can be stabilized to maintain the MBCCF structure. The theoretical results were further supported by illustrative examples, confirming the practicality and effectiveness of the proposed approach.

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