On the Estimation of Global Network Topology Information

Thomas Van Assche, Michael Defoort, Giordano Pola and Mohamed Djemai

***Abstract*— Control schemes for multi-agent systems are often designed to be decentralized to increase their fault robustness and reduce computational costs compared to a centralized scheme. The literature on consensus problems and, more generally, decentralized control is extensive. However, most work assumes knowledge of quantities related to the communication topology. These quantities are centralized because each agent does not have access to them solely through its own information and that of its neighbors. Therefore, it is necessary to estimate them in a distributed manner to obtain fully decentralized control laws. In this paper, the problem is detailed and illustrated with examples. A review of existing methods for estimating the Laplacian matrix of a graph or the number of agents in the network is then presented. Some interesting research directions are also discussed.**

***Index Terms*— Multi-agent systems, Distributed estimation, Laplacian matrix.**

I. INTRODUCTION

Networked systems are used in many applications [1]. Sensor networks monitor physical or environmental conditions such as temperature, sound, vibration, or pressure better than isolated sensors. Indeed, the network can achieve an accurate estimate of the variables of interest with a lower computational cost by exchanging local information with neighbors. Molecular devices can be combined to create a molecular dynamical system. The latter produces chemical reactions used for transitioning between several stable states of a molecule. The structure of social networks is also studied in theoretical sociology, organizational studies, and sociolinguistics, as well as the impact of the dynamic of an agent on the network. Complex networks of heterogeneous power system elements are often used to deliver energy from generators to machines. In particular, smart grids have been widely studied for using energy in a more efficient manner.

Multi-agent systems usually pursue a common objective or distributively complete a task using local data exchanges. Each agent is only aware of its own information and that of its neighbors. The communications among agents are represented using graph theory, and the set of edges in the graph describes the communication topology. An important element related to this topology is the Laplacian matrix, because it gives information about the connectivity of the network [2]. In fact, the set of neighbors of each agent and the weight of each communication link are directly obtained from the Laplacian matrix. Furthermore, its spectrum is often involved in the parameterization of control laws for multi-agent systems [3], [4], [5]. The smallest non-zero eigenvalue of the Laplacian matrix describing an undirected graph, called algebraic connectivity, is particularly important as it is a crucial variable used for connectivity maintenance [6], [7] or network analysis, especially the study of controllability, observability, or connectivity [8]. The number of agents present in the system is also a global variable that each agent does not necessarily know, especially in the case of open multi-agent systems, i.e. networked systems with a variable number of nodes.

To obtain a completely decentralized scheme, it is necessary to distributively estimate any topological information, such as the spectrum of the Laplacian matrix or the number of agents.

The rest of the paper is organized as follows. Some important graph-theoretic concepts are summarized and the problem is introduced and illustrated in Section 2. In Section 3, a review of some methods for estimating topological information is presented, i.e. the spectrum of the Laplacian matrix, specifically the algebraic connectivity, and the number of agents. Section 4 proposes some directions for deepening the research concerning the decentralized estimation of global topological quantities.

Notations: Symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ denote the sets of real numbers, positive real numbers, and natural numbers, respectively. Vectors are written in bold, and matrices in capital letters. As examples, $\mathbf{1} = [1, \ldots, 1]^T$, $\mathbf{0} = [0, \ldots, 0]^T \in \mathbb{N}^N$, I_N is the identity matrix of N, and 0_N is the matrix of size N filled with zero entries, with $N \in \mathbb{N}$. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and the greatest eigenvalues of a square matrix A, respectively. A^* is the conjugate transpose of matrix A. For $p \in \mathbb{N} \cup \{\infty\}$, $\|\mathbf{u}\|_p$ denotes the p-norm of a vector $\mathbf{u} \in \mathbb{R}^N$. Given a finite set X, symbol $card(X)$ denotes its cardinality.

II. PRELIMINARIES AND PROBLEM FORMULATION

Some graph-theoretical concepts will be recalled. A communication network is represented by a graph denoted $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{v_1, \ldots, v_N\}$ the set of nodes (agents) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges whose elements are written as $e_{ij} = \{v_i, v_j\}$. The graph can be weighted and denoted

Thomas Van Assche and Michael Defoort are with the Department of Automatic Control, LAMIH UMR CNRS8201, Université Polytechnique Hauts-de-France, France. Thomas.VanAssche@uphf.fr, michael.defoort@uphf.fr

Thomas Van Assche and Giordano Pola are with the Department of Engineering, Information Science and Mathematics, Center of Excellence DEWS, Università degli Studi dell'Aquila, Italy. thomas.vanassche@graduate.univaq.it, giordano.pola@univaq.it

Mohamed Djemai is with the LÉA Laboratory, ENSEA, Cergy, France, and the Department of Electronic and Communications Engineering, Kuwait College of Science and Technology, Kuwait mohamed.djemai@ensea.fr

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, where $w : \mathcal{E} \to \mathbb{R}^+$ is a function associating a weight to each edge. The set of out-neighbors $\mathcal{N}_i^+ = \{v_j \in \mathcal{V} \,|\, (v_i, v_j) \in \mathcal{E}\}$ contains the $N_i^+ = card(\mathcal{N}_i^+) \leq N$ agents to which the agent i sends information. The set of in-neighbors $\mathcal{N}_i^- = \{v_j \in \mathcal{V} \,|\, (v_j, v_i) \in \mathcal{E}\}$ contains the $N_i^- = card(\mathcal{N}_i^-) \leq N$ agents from which the agent i receives information.

The graph $\mathcal{G}$ is *undirected* if $\forall i, j \in \{1, \ldots, N\},\ e_{ij} = e_{ji}$. Otherwise, the graph is *directed*. For an undirected graph, $\mathcal{N}_i^- = \mathcal{N}_i^+ = \mathcal{N}_i$ and $N_i^- = N_i^+ = N_i$.

A *path* is a sequence of edges of the form $e_{ij}, e_{jk}, \ldots$. A directed graph is said to be *strongly connected* if there is a path from every node to every other node. An undirected graph is called *connected* analogously. A directed graph is *weakly connected* if it is connected when the edge e_{ji} is added to the set $\mathcal{E}$ for every $e_{ij} \in \mathcal{E}$.

The *adjacency matrix* represents the communication network and is defined as $A \in \mathbb{R}^{N\times N}$ with $a_{ij} = w(e_{ij}) > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. If the graph representing the network is undirected, $a_{ij} = a_{ji}$. If the graph is unweighted, $w(e_{ij}) = 1$ for every $(i, j) \in \mathcal{E}$. The *degree matrix* $D \in \mathbb{R}^{N\times N}$ with entries $d_{ii} = \sum_{j=1}^{N} a_{ij}$ is a diagonal matrix. The *Laplacian matrix* associated with graph $\mathcal{G}$ is defined by $L = D - A$. L is a positive semidefinite matrix and its eigenvalues are denoted $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. The Laplacian matrix of an undirected graph is symmetric and $0 = \lambda_1 < \lambda_2$ if the graph is connected, with λ_2 the *algebraic connectivity* and $\mathbf{v_2}$ an associated eigenvector whose elements are denoted $v_{2_1}, \ldots, v_{2_N}$;

To demonstrate the purpose of this survey, some examples of control laws for multi-agent systems are considered. In [3], a weighted connected undirected graph is considered whose nodes are agents with double-integrator dynamics:

$$\begin{aligned}
\dot{\xi}_i &= \zeta_i, \\
\dot{\zeta}_i &= u_i, \\
u_i &= \sum_{v_j \in \mathcal{N}_i} a_{ij}(\gamma_0(\xi_j - \xi_i) + \gamma_1(\zeta_j - \zeta_i)),
\end{aligned}$$

where $\xi = [\xi_1, \ldots, \xi_N]^T \in \mathbb{R}^N$, $\zeta = [\zeta_1, \ldots, \zeta_N]^T \in \mathbb{R}^N$ and $u_i \in \mathbb{R}$. As $\gamma_0 > 0$ is given, the maximum consensus speed is achieved as

$$\gamma_1 = \frac{2\sqrt{\gamma_0 \lambda_N}}{\sqrt{\lambda_2(2\lambda_N - \lambda_2)}}.$$

In [4], the authors consider a leader-follower consensus problem. The dynamics of each follower and of the leader are given by

$$\begin{aligned}
\dot{\mathbf{x}}_\mathbf{i} &= f(t, \mathbf{x_i}) + \mathbf{u_i}, \quad i = 1, \ldots, N, \\
\dot{\mathbf{x}}_\mathbf{0} &= f(t, \mathbf{x_0}) + \mathbf{u_0},
\end{aligned}$$

where $\mathbf{x_0} \in \mathbb{R}^m$ is the virtual leader state, $\mathbf{u_0}$ is its control input satisfying $\|\mathbf{u_0}\| \leq l$, $\mathbf{x_i} \in \mathbb{R}^m$ is the state of the agent v_i, $\mathbf{u_i}$ is its control input, and the continuous in t function $f : \mathbb{R}^+ \times D \to \mathbb{R}^m$ is the uncertain nonlinear dynamics of the agents, $D \subset \mathbb{R}^m$ being a domain containing the origin. Furthermore, there exist positive constants l_1, l_2, l_3 such that $\forall \mathbf{x_1}, \mathbf{x_2} \in D,\ \forall t \geq 0$

$$\|f(t, \mathbf{x_1}) - f(t, \mathbf{x_2})\|^2 \leq l_1 + l_2\|\mathbf{x_1} - \mathbf{x_2}\|^2 + l_3\|\mathbf{x_1} - \mathbf{x_2}\|^4.$$

The communication network among followers is represented by a connected undirected weighted graph. The matrix $H = L + diag(a_{10}, \ldots, a_{N0})$ describes the entire communication topology, where $a_{i0} > 0$ if the leader state is available to the i-th agent and $a_{i0} = 0$ otherwise. If at least one $a_{i0} > 0$, then the consensus is achieved with the control protocol

$$\begin{aligned}
\mathbf{u_i} &= \alpha \left\lfloor \sum_{j=0}^{N} a_{ij}(\mathbf{x_j} - \mathbf{x_i}) \right\rceil^2 \\
&\quad + \beta \sum_{j=0}^{N} a_{ij}(\mathbf{x_j} - \mathbf{x_i}) + \gamma\ sign\left(\sum_{j=0}^{N} a_{ij}(\mathbf{x_j} - \mathbf{x_i})\right),
\end{aligned}$$

where the function $\lfloor a \rceil^b$ is described as $\lfloor a \rceil^b = |a|^b sign(a)$, and the parameters are defined as

$$\begin{aligned}
\alpha &= \frac{\sqrt{l_3}(\lambda_{max}(H))^{3/2}}{(\lambda_{min}(H))^{7/2}} + \frac{\epsilon\sqrt{N}}{(2\lambda_{min}(H))^{3/2}}, \\
\beta &= \frac{\sqrt{l_2}}{\lambda_{min}(H)}, \\
\gamma &= l + \sqrt{l_1 N} + \epsilon\sqrt{\frac{\lambda_{max}(H)}{2\lambda_{min}(H)}},
\end{aligned}$$

with $\epsilon > 0$.

In [5], the authors consider a connected weighted undirected graph representing a network of N agents whose dynamics are described by

$$\dot{\mathbf{x}}_\mathbf{i}(t) = A\mathbf{x_i}(t) + B\mathbf{u_i}(t) + f(\mathbf{x_i}),\ i = 1, \ldots, N,$$

where $\mathbf{x_i} \in \mathbb{R}^p$ is the state of the agent v_i, $\mathbf{u_i} \in \mathbb{R}^q$ is its control input, $f(.) : \mathbb{R}^p \to \mathbb{R}^p$ is a nonlinear function, and $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times q}$ express the matrices of the system. The following event-triggered control protocol is designed to achieve consensus:

$$\mathbf{u_i}(t) = c\mathcal{K} \sum_{j=1}^{N} a_{ij}\left(\tilde{\mathbf{x}}_\mathbf{i}(t) - \tilde{\mathbf{x}}_\mathbf{j}(t)\right),\ i = 1, \ldots, N,$$

where $\mathcal{K}$ is a feedback matrix, $c > 0$ satisfies $c \geq \frac{\xi}{\lambda_2}$ with $\xi > 0$, and $\tilde{\mathbf{x}}_\mathbf{i}(t) = e^{A(t - t_k^i)}\mathbf{x_i}(t_k^i)$ is the state estimation of $\mathbf{x_i}(t)$ for the i-th agent, with $t_k^i\,(t_0^i = 0)$ the k-th triggering instant of the i-th agent.

From these three examples, one can see that the information related to the communication topology is necessary for defining the parameters of decentralized control laws for multi-agent systems. Therefore, in the following, techniques for distributively estimating the spectrum of the Laplacian matrix and the number of agents interacting in the network are going to be reviewed, as well as schemes designed specifically to estimate the algebraic connectivity.

III. Distributed estimation of topological information

A. Spectrum of the Laplacian matrix

Some studies aim to estimate the entire spectrum of the Laplacian matrix. These are powerful because some control laws are parametrized depending on different eigenvalues of the topology matrix. Indeed, most of those works estimate the eigenvalues of a matrix related to the graph but not necessarily the Laplacian one. In the following, the notation W is used to refer to a matrix describing the communication topology, for instance the adjacency matrix or the Laplacian matrix. In [9], the topology matrix W representing a strongly connected directed graph is transformed into a nonsingular matrix $\bar{W}$, and the eigenvalues of W can be inferred from those of $\bar{W}^{-1}$. The principle is then to compute $\bar{W}^{-1}$ distributively using a consensus-based update law to make an estimator matrix converge to this inverse. The main drawback of this method is its computational cost because each agent needs to store N^2 values.

More works focus on using observations of the update law. The authors of [10] consider $K_i + 1$ observations of a linear update law evolving under a connected unweighted undirected graph, where K_i is a quantity related to the observability for the agent v_i. It can be upper-bounded by $N - N_i$. The observations are then gathered to create a matrix whose left singular vectors can be retrieved. From these vectors, it is possible to compute the eigenvalues of the topology matrix. The estimation is precise, but depending on the connectivity of the graph, it is possible that some nodes cannot observe the eigenvalues. Also, relying on observations requires the state to be updated a certain number of times before starting the procedure of spectrum estimation.

The observations can also be used together with optimization problems. In [11], a cost function is proven to be minimal only if all distinct eigenvalues of the Laplacian matrix can be retrieved from the set of all step sizes used during optimization. The authors propose an algorithm that tests each step size to determine if it is linked to one of the distinct eigenvalues of L. The output of this algorithm is a set from which all distinct eigenvalues can be inferred. The approach of [12] relies on using the discretized wave equation as the update law for the agents. From the history of states, a data-driven model is identified by solving an optimization problem in a distributed manner. A polynomial is then constructed using the optimizing quantities as coefficients, and the eigenvalues of the normalized Laplacian matrix can be inferred from its roots.

In the same idea as [12], some studies aim to create a polynomial whose roots are related to the eigenvalues of the topology matrix. The following dynamics are considered in [13]:

$$\mathbf{x}(k+1) = W\mathbf{x}(k),$$

where W is the topology matrix associated with a connected weighted undirected graph. As the system is discrete, one can write $\mathbf{x}(N) = W^N\mathbf{x}(0)$ and use the Cayley-Hamilton theorem

$$W^N = -y^{(0)} I_N - y^{(1)} W - \cdots - y^{(N-1)} W^{N-1},$$

where $y^{(i)}$ is the i-th coefficient of the characteristic polynomial of W. Therefore, the following equalities hold:

$$\begin{aligned}
\mathbf{x}(N) &= (-y^{(0)} I_N - y^{(1)} W - \cdots - y^{(N-1)} W^{N-1})\mathbf{x}(0),\\
-\mathbf{x}(N) &= y^{(0)}\mathbf{x}(0) + y^{(1)}\mathbf{x}(1) + \cdots + y^{(N-1)}\mathbf{x}(N-1),
\end{aligned}$$

$$\begin{bmatrix} -x_1(N) \\ -x_2(N) \\ \vdots \\ -x_N(N) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_1(1) & \dots & x_1(N-1) \\ x_2(0) & x_2(1) & \dots & x_2(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ x_N(0) & x_N(1) & \dots & x_N(N-1) \end{bmatrix} \begin{bmatrix} -y^{(0)} \\ -y^{(1)} \\ \vdots \\ -y^{(N-1)} \end{bmatrix},$$

$$\mathbf{b} = X\mathbf{y}^*.$$

If the matrix X is nonsingular, then the multi-agent system is able to cooperatively find $\mathbf{y}^*$ such that the previous equality holds by using the following dynamics:

$$\dot{\mathbf{y}}_\mathbf{i}(t) = -\alpha_i(\mathbf{X_i}^T\mathbf{y_i}(t) - b_i)\mathbf{X_i} - \sum_{j\in\mathcal{N}_i} \beta_{ij}(\mathbf{y_i}(t) - \mathbf{y_j}(t)),$$

where $\mathbf{y_i}(t) = [y_i^{(0)}(t), y_i^{(1)}(t), \dots, y_i^{(N-1)}(t)]^T$ is the estimate of $\mathbf{y}^*$ of the agent v_i, $\mathbf{X_i}^T$ is the i-th row of the matrix X, b_i is the i-th element of $\mathbf{b}$, and $\alpha_i > 0 \quad \forall v_i \in \mathcal{V}$ and $\beta_{ij} > 0 \quad \forall \{i,j\} \in \mathcal{E}$ are parameters. $\mathbf{y_i}(t)$ exponentially converges to $\mathbf{y}^*$. Therefore, each agent can continuously estimate the roots of the following polynomial, i.e. the spectrum of W:

$$\begin{aligned}
det(\lambda I_N - W) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_N)\\
&= \lambda_N + y^{(N-1)}\lambda^{N-1} + \cdots + y^{(1)}\lambda + y^{(0)}.
\end{aligned}$$

The authors also propose a method to perturb the matrix W to almost ensure that the matrix X is nonsingular. This paper presents a rather simple method, illustrating some similar works that can be found in the literature, such as in [14] that is based on the minimal polynomial of a matrix.

The drawback of these studies is often that the proposed method is not scalable by definition: if the goal is to estimate all the eigenvalues of a matrix, then if the number of agents increases, the size of the matrix and the number of its eigenvalues do so. Indeed, the scalability of the communication in Table I should be considered together with the scalability of the computational cost and the scalability of the memory storage. The latter is difficult to properly quantify. For instance, in [10], [11], the maximum number of observations increases linearly with the size of the network, but the minimum number of observations needed is not studied. In [14], the storage of a matrix is needed. The maximum size of the matrix is $N \times N$, but the minimum necessary size is not given. The scalability of the computational cost is also difficult to quantify because it depends on the method that is used. For example, a root-finding algorithm is needed in [12], [13], and the computational cost depends on the algorithm that is considered. It is also hard to estimate the cost of the computation of the left singular vector of a matrix [10] or of the kernel of a matrix [14].

TABLE I

COMPARISON OF THE PRESENTED METHODS FOR THE ESTIMATION OF THE SPECTRUM OF THE TOPOLOGY MATRIX.

Reference	Communication scalability	Decentralization	Convergence
[9]	$O(N^2)$	Fully decentralized	Exponential
[10]	$O(1)$	Fully decentralized	Finite-time
[11]	$O(1)$	Need for the final consensus value	Linear
[12]	$O(1)$	Fully decentralized	Asymptotic
[13]	$O(1)$	Fully decentralized	Exponential
[14]	$O(1)$	Fully decentralized	Finite-time

B. Algebraic connectivity

As the algebraic connectivity λ_2, among the whole spectrum of the Laplacian matrix, is playing a particularly important role for network analysis and connectivity maintenance, some studies focus on estimating this particular eigenvalue. In [15], the authors propose to estimate a lower and an upper bound to λ_2, written $\check{\lambda}_2$ and $\hat{\lambda}_2$, respectively. The error between the actual value of λ_2 and the bounds $\check{\lambda}_2$ and $\hat{\lambda}_2$ is bounded. This method relies on exploiting the role of λ_2 in achieving consensus by distributively computing the discrepancy between the state of the agents. Although this method works with weakly connected directed graphs, obtaining a set of possible values may not be precise enough depending on the application.

Most studies are based on the power iteration algorithm. Consider a diagonalizable matrix A and a vector $\mathbf{b_k}$. The sets $\{\lambda_1, \ldots, \lambda_n\}$ and $\{v_1, \ldots, v_n\}$ are the sets of eigenvalues and associated eigenvectors of A, respectively, with λ_n the greatest eigenvalue of the set. If $\mathbf{b_0}$ has a nonzero component in the direction of v_n, then

$$\mathbf{b_{k+1}} = \frac{A\mathbf{b_k}}{\|A\mathbf{b_k}\|},$$
$$\mu_k = \frac{\mathbf{b_k}^* A\mathbf{b_k}}{\mathbf{b_k}^*\mathbf{b_k}},$$

exponentially converge to v_n and λ_n, respectively. As $-\lambda_2$ is the second greatest eigenvalue of $-L$, this algorithm is relevant to estimate it. In fact, it is possible to construct a matrix whose greatest eigenvalue is λ_2 by using a deflation technique that computes a matrix whose undesired eigenvalues become 0.

In [16], the authors deflate the Laplacian matrix and then use a decentralized modified version of the power iteration algorithm to acquire information about λ_2 through a vector similar to $\mathbf{b_k}$. The algebraic connectivity can be estimated from the relation between the last three instances of $\mathbf{b_k}$. This method is computationally expensive because it requires $2N$ decentralized estimators and an algorithm to solve a set of linear equations. However, it works with strongly connected directed graphs and can estimate both a real and complex eigenvalue. The method presented in [17] has the same advantages and is more scalable. Indeed, only four estimators are used to decentralize the proposed generalized power iteration algorithm.

A continuous-time version of the power iteration algorithm has been developed in [18] and has been the basis of other studies for its simplicity, scalability, and efficiency. The authors consider the following dynamics:

$$\dot{\mathbf{x}} = -\frac{k_1}{N}\mathbf{1}\mathbf{1}^T\mathbf{x} - k_2 L\mathbf{x} - k_3\left(\frac{\mathbf{x}^T\mathbf{x}}{N} - 1\right)\mathbf{x}, \qquad (1)$$

where $\mathbf{x} = [x_1, \ldots, x_N]^T$ is the state of the estimator for all agents and k_1, k_2, k_3 are positive constants. The first term of (1) prevents the model from converging to the eigenvector $\mathbf{v_1} = \mathbf{1}$ associated with $\lambda_1 = 0$; it is similar to a deflation. The second one makes the model converge to an eigenvector $\mathbf{v_2}$ associated with λ_2 which is considered the leading eigenvalue because of the first term. The third term stabilizes the norm of the estimation. As long as $\mathbf{v_2}\mathbf{x}(0) \neq 0$ and the gains satisfy

$$k_1 > \lambda_2 k_2, \qquad k_3 > \lambda_2 k_2,$$

x converges to an eigenvector $\bar{\mathbf{v}}_\mathbf{2}$ of L associated with λ_2 and satisfying $\|\bar{\mathbf{v}}_\mathbf{2}\| = \sqrt{N\frac{k_3 - k_2\lambda_2}{k_3}}$. This scheme is centralized due to terms $\mathbf{1}^T\mathbf{x}$ and $\mathbf{x}^T\mathbf{x}$ which require that each agent knows all x_i. As these quantities are averages, the authors propose to use PI average consensus estimators to estimate them in a decentralized manner:

$$\dot{z}_i = \gamma(\alpha_i - z_i) - K_P \sum_{j\in\mathcal{N}_i}(z_i - z_j) + K_I \sum_{j\in\mathcal{N}_i}(w_i - w_j),$$
$$\dot{w}_i = -K_I \sum_{j\in\mathcal{N}_i}(z_i - z_j).$$

All α_i converge to $\frac{\sum_{i=1}^N \alpha_i}{N}$. In the context of this article, two estimators are needed and $\alpha_i^{(1)}, \alpha_i^{(2)}$ are, respectively, x_i and x_i^2. Using these estimators, a decentralized and continuous version of the power iteration algorithm is obtained. As $\mathbf{x}$ converges to $\bar{\mathbf{v}}_\mathbf{2}$, it is possible for each agent to compute an estimate λ_{2_i} of λ_2 as follows:

$$\lim_{t\to\infty} z_i^{(2)} = \frac{\|\mathbf{x}\|_2^2}{N} = \frac{k_3 - k_2\lambda_2}{k_3},$$
$$\lambda_{2_i} = \frac{k_3}{k_2}(1 - z_i^{(2)}).$$

In [19], faster estimators are proposed, improving the efficiency of the method. However, these estimators converge asymptotically, which implies a perpetual error of estimation in the centralized scheme. To ensure that the latter is retrieved and that the convergence is ensured, the authors of [20] use estimators that converge in finite-time. Other centralized continuous versions of the power iteration algorithm are proposed in [21], [22]. Following the same principle, some estimators are used to approximate the centralized terms of the schemes considered.

TABLE II
COMPARISON OF THE PRESENTED METHODS FOR THE ESTIMATION OF THE ALGEBRAIC CONNECTIVITY.

Reference	Scalability	Decentralization	Convergence
[15]	$O(1)$	Fully decentralized	Asymptotic convergence of bounds. The final distance between the bounds is bounded.
[16]	$O(N)$	Fully decentralized	Polynomial
[17]	$O(1)$	Need for another method for the computation of a left eigenvector of L	Asymptotic
[18]	$O(1)$	λ_2 in the parametrization	Asymptotic
[19]	$O(1)$	λ_2 in the parametrization	Asymptotic
[20]	$O(1)$	λ_2 in the parametrization	Asymptotic
[21]	$O(1)$	Fully decentralized	Asymptotic
[22]	$O(1)$	Fully decentralized	Asymptotic

In this section, the scalability is better than in the previous one, because only one value is estimated regardless of the size of the network. Moreover, none of the presented methods requires the history of the states or a specific algorithm that has an additional computational cost. Thus, the column "Scalability" in Table II represents the general scalability of the method, taking into account the communication, the memory storage and the computational cost.

C. Number of agents

In addition to the connectivity of the graph, the number of vertices is also a global information that an individual agent does not necessarily possess, especially if it varies over time such as in the case of open multi-agent systems. To the best of my knowledge, most existing studies are based on statistical methods and are not necessarily decentralized. For example, the capture-recapture method presented in [23] considers a network of size N among which M nodes receive a seed. N' nodes are then queried, and M' nodes had received a seed among them. It is expected that $\frac{M}{N} = \frac{M'}{N'}$, from which N can be computed. Another example is the random walk method presented in [24]. The most simple version of this method defines an initial node v_i with a counter value $X = \frac{\phi(i)}{d_i}$ where $\phi(i) = 1$ for all i and d_i is the number of neighbors of the node v_i. This node sends its number i and its value X to a neighbor v_j that increments X by $\frac{\phi(j)}{d_j}$. This process is repeated until node i receives the message and the latter computes the estimate of the system size $\hat{\phi} = d_i X$. This version can be derived to obtain a continuous-time version of the random walk method that is based on the time elapsed before the information returns to the initial node. This type of method is designed for the analysis of large scale networks. Therefore, it is not decentralized and is significantly less efficient for smaller systems.

The method presented in [25] has a greater level of decentralization. The authors consider a scheme that converges to the number of agents N only if a parameter k satisfies $k > N^3$. The dynamics of every agent are the same except for one of them. The two problems are to choose the one that has different dynamics in a distributed manner and to set k properly. The proposed solution is to give each agent a unique natural identifier and estimate the greatest. Indeed, the largest identifier is at least equal to N, which is sufficient to set k, and each agent can set the dynamics of its network size estimator depending on its identifier and the estimated largest. This method is scalable and works for smaller systems. The problem of giving each agent a unique identifier can be solved by taking the one of the communication device, i.e. the MAC addresses.

In [26], the general update law $\mathbf{x}(k+1) = P\mathbf{x}(k)$ is considered, where P is a column stochastic matrix whose entries are positive and smaller than 1. Each agent v_i chooses the values of the i-th row of P. Based on this update law, the two following expressions are derived:

$$y_j(k+1) = p_{jj}y_j(k) + \sum_{i\in\mathcal{N}_j^-} p_{ji}y_i(k),$$
$$z_j(k+1) = p_{jj}z_j(k) + \sum_{i\in\mathcal{N}_j^-} p_{ji}z_i(k).$$

If the directed graph representing the considered network is strongly connected, and if $\mathbf{y}(0) = \mathbf{1}$ and all $z_j(0) = 0$ except one $z_j(0) = 1$, then

$$\lim_{k\to\infty} \frac{z_j(k)}{y_j(k)} = \frac{\sum_{i=1}^N z_i(0)}{\sum_{i=1}^N y_i(0)} = \frac{1}{N}. \tag{2}$$

To satisfy the initial condition of $\mathbf{z}$, the authors propose that each agent sets $z_j(0) = 1$ and randomly initializes another variable $w_j(0)$. The latter is updated according to a max-consensus algorithm. The first time $w_j(k+1) \neq w_j(k)$, the agent v_j subtracts 1 from $z_j(k+1)$. When the max-consensus is achieved, the property (2) is retrieved. This method is also derived for asynchronous communication. This work is scalable, but no method for setting the weights of the column-stochastic matrix P is clearly presented.

IV. CONCLUSION AND DISCUSSION

Several methods already exist for distributively estimating topological information. However, many research directions are interesting to extend the studies that have been presented.

For instance, many works that consider directed communication links assume that the network is strongly connected, which is a strong assumption. The weakest assumption is that the graph is weakly connected. Indeed, a graph that does not satisfy this condition contains an isolated node which can be considered outside the system as it cannot update its information according to the rest of the agents.

TABLE III

COMPARISON OF THE PRESENTED METHODS FOR THE ESTIMATION OF THE NUMBER OF AGENTS.

Reference	Scalability	Decentralization	Convergence
[23]	$O(N)$ for the central unit	Centralized	Not formally analyzed
[24]	$O(N)$ for the central unit	Centralized	Not formally analyzed
[25]	$O(1)$	Fully decentralized	Finite time estimation of N with exponential rate by rounding
[26]	$O(1)$	Need for a column stochastic topology matrix	Asymptotic

Another possible research direction is to design scalable estimation methods. In fact, most of the works presented in this paper are decentralized, but the computational cost of some becomes heavier as the number of agents in the system increases. This is a problem for many applications with a large number of agents, as one of the goals of a decentralized approach is to make the system scalable.

The most important direction is the dynamic communication topology. All of the studies considered in this survey consider a constant graph, but in reality communication links may be broken or created during the execution, and weights of each link can vary over time depending on the quality of the communication between two agents, for example. The exchange of information might also be delayed or asynchronous. These cases are common in a realistic context and, therefore, must be studied in the future.

V. ACKNOWLEDGMENTS

This work has been partially supported by the CNRS through the MITI interdisciplinary programs through the exploratory research program, by the ANR agency under TROCOOL project (ANR-25-CE10-4393), by the Center of Excellence DEWS, University of L'Aquila, Italy, and by the Italian Ministry of Economic Development, EADAPTIVE Project N. F/310261/01-02/X56.

We thank the Professor Giulia De Masi from the Sorbonne University, Abu Dhabi, for useful discussion about this paper.

REFERENCES

[1] M. Mesbahi and M. Egerstedt, Graph theoretic methods in multiagent networks, Princeton University Press. Princeton and Oxford, 2010.

[2] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra and its Applications, vol. 197, no. 198, pp. 143-176, 1994.

[3] J. Zhu, On consensus speed of multi-agent systems with double-integrator dynamics, Linear Algebra and its Applications, vol. 434, no. 1, pp. 294-306, Jan. 2011.

[4] M. Defoort, A. Polyakov, G. Demesure, M. Djemai, and K. Veluvolu, Leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics, IET Control Theory & Applications, vol. 9, no. 14, pp. 2165-2170, 2015.

[5] J. Zhang, H. Zhang, Y. Cai, and W. Wang, Consensus control for nonlinear multi-agent systems with event-triggered communications, Applied Mathematics and Computation, vol. 408, p. 126341, Nov. 2021.

[6] M. C. D. Gennaro and A. Jadbabaie, Decentralized control of connectivity for multi-agent systems, in 45th IEEE Conference on Decision and Control (CDC), 2007.

[7] M. M. Zavlanos and G. Pappas, Potential fields for maintaining connectivity of mobile networks, IEEE Transactions on Robotics, vol. 23, no. 4, pp. 812-816, 2007.

[8] M. Franceschelli, S. Martini, M. Egerstedt, A. Bicchi, and A. Giua, Observability and controllability verification in multi-agent systems through decentralized Laplacian spectrum estimation, in 49th IEEE Conference on Decision and Control (CDC), Dec. 2010, pp. 5775-5780.

[9] A. Gusrialdi and Z. Qu, Distributed estimation of all the eigenvalues and eigenvectors of matrices associated with strongly connected digraphs, IEEE Control Systems Letters, vol. 1, no. 2, pp. 328-333, Oct. 2017.

[10] A. Kibangou and C. Commault, Decentralized Laplacian Eigenvalues Estimation and Collaborative Network Topology Identification, in NecSys 2012 - 3rd IFAC Workshop on Distributed Estimation and Control in Networked Systems, Santa Barbara, United States, Sep. 2012.

[11] T.-M.-D. Tran and A. Y. Kibangou, Distributed estimation of Laplacian eigenvalues via constrained consensus optimization problems, Systems & Control Letters, vol. 80, pp. 56-62, Jun. 2015.

[12] D. Deplano, C. Congiu, A. Giua, and M. Franceschelli, Distributed estimation of the Laplacian spectrum via wave equation and distributed optimization, IFAC-PapersOnLine, vol. 56, no. 2, pp. 6952-6957, 2023.

[13] M. Yang and C. Y. Tang, Distributed estimation of graph spectrum, in American Control Conference (ACC), Jul. 2015, pp. 2703-2708.

[14] T. Charalambous, M. G. Rabbat, M. Johansson, and C. N. Hadjicostis, Distributed finite-time computation of digraph parameters: left-eigenvector, out-degree and spectrum, IEEE Transactions on Control of Network Systems, vol. 3, no. 2, pp. 137-148, Jun. 2016.

[15] C. Li, Z. Qu, D. Qi, and F. Wang, Distributed finite-time estimation of the bounds on algebraic connectivity for directed graphs, Automatica, vol. 107, pp. 289-295, Sep. 2019.

[16] H. A. Poonawala and M. W. Spong, Decentralized estimation of the algebraic connectivity for strongly connected networks, in American Control Conference (ACC), Jul. 2015, pp. 4068-4073.

[17] M. Mehdi Asadi, Mohammad Khosravi, Hesam Mosalli, Stephane Blouin, and Amir G. Aghdam, Generalized Power Iteration with Application to Distributed Connectivity Estimation of Asymmetric Networks, Aug. 08, 2023.

[18] P. Yang, R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, Decentralized estimation and control of graph connectivity for mobile sensor networks, Automatica, vol. 46, no. 2, pp. 390-396, Feb. 2010.

[19] Y. Zhang, S. Li, and J. Weng, Distributed estimation of algebraic connectivity, IEEE Transactions on Cybernetics, vol. 52, no. 5, pp. 3047-3056, May 2022.

[20] T. Van Assche, M. Defoort, G. Pola, M. Djemai, Distributed estimation of the algebraic connectivity for undirected graphs, 23rd IFAC World Congress, 2026.

[21] I. Malli, C. P. Bechlioulis, and K. J. Kyriakopoulos, Robust distributed estimation of the algebraic connectivity for networked multi-robot systems, in IEEE International Conference on Robotics and Automation (ICRA), 2021, pp. 9155-9160.

[22] P. Di Lorenzo and S. Barbarossa, Distributed Estimation and Control of Algebraic Connectivity Over Random Graphs, IEEE Transactions on Signal Processing, vol. 62, no. 21, pp. 5615-5628, Nov. 2014.

[23] S.-L. Peng, S.-S. Li, X.-K. Liao, Y.-X. Peng, and N. Xiao, Estimation of a Population Size in Large-Scale Wireless Sensor Networks, J. Comput. Sci. Technol., vol. 24, no. 5, pp. 987-997, Sep. 2009.

[24] L. Massoulie, E. Le Merrer, A.-M. Kermarrec, and A. J. Ganesh, Peer counting and sampling in overlay networks: random walk methods, in PODC '06: Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing, in PODC '06. New York, NY, USA: Association for Computing Machinery, Jul. 2006, pp. 123-132.

[25] D. Lee, T. Kim, S. Lee, and H. Shim, Fully Decentralized Design of Initialization-free Distributed Network Size Estimation, Jan. 15, 2024.

[26] I. Shames, T. Charalambous, C. N. Hadjicostis, and M. Johansson, Distributed network size estimation and average degree estimation and control in networks isomorphic to directed graphs, in 2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Oct. 2012, pp. 1885-1892.