A learning-enhanced set-membership estimator with zonotopic guarantees

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Abstract— **This paper proposes a hybrid state estimation framework combining Model-based prediction, a neural correction mechanism and constrained zonotope-based uncertainty propagation. The proposed approach leverages a recurrent neural network to learn residual dynamics and compensate for model inaccuracies, while preserving the structure and interpretability of the underlying physical model. In parallel, uncertainty is represented and propagated using constrained zonotopes, enabling the computation of guaranteed bounds on the state estimate under bounded disturbances. The proposed method relies on a decomposition between the learned state estimate (center) and a set-based representation of the estimation error, which is propagated through a Jacobian-based linearization of the neural model. This structure allows for the integration of data-driven adaptability with formal set-membership guarantees. The effectiveness of the proposed framework is demonstrated on a benchmark system, showing improved estimation accuracy compared to model-based approaches while maintaining reliable uncertainty envelopes.**

I. INTRODUCTION

Robust state estimation is a fundamental problem in control and monitoring of dynamic systems subject to uncertainty. In many real-world applications, disturbances affecting both measurements and system dynamics are difficult to characterize accurately, which challenges reliable state reconstruction. Classical approaches rely on different uncertainty modeling paradigms, each offering specific advantages and limitations. Probabilistic methods, notably the Kalman filter and its extensions, are widely used due to their optimality under Gaussian assumptions and computational efficiency [1]. However, their performances strongly depend on the validity of statistical assumptions on noise distributions and model accuracy. In practice, uncertainties are often non-Gaussian, time-varying, or poorly characterized, which can significantly degrade estimation performance. As an alternative, set-membership approaches model uncertainties as unknown but bounded quantities, enabling the computation of guaranteed bounds on the state estimate without relying on probabilistic assumptions [2], [3], [4]. Zonotopic and constrained zonotopic representations enable efficient uncertainty propagation in linear systems [5], but may introduce conservatism, especially under nonlinearities or overestimated bounds. In parallel, learning-based methods, particularly recurrent neural networks such as GRU (Gated Recurrent Units) and LSTM (Long Short-Term Memory) have shown strong ability to capture complex dynamics from data [6], [7], [8]. These approaches can significantly improve estimation accuracy in the presence of model mismatch or unmodeled dynamics. Nevertheless, purely data-driven methods lack interpretability and, more importantly, do not provide formal guarantees on the estimation error, which limits their applicability in safety-critical contexts. To address these limitations, recent research has explored hybrid approaches that combine model-based and learning-based techniques. For instance, learning-enhanced Kalman filtering methods incorporate neural networks to compensate for modeling errors while preserving the recursive estimation structure [9], [10]. Other works have investigated the integration of probabilistic and set-membership frameworks to balance statistical efficiency and robustness [11]. More recently, neuro-symbolic approaches combining neural networks with set-based representations have been proposed to provide robustness guarantees while leveraging the expressive power of deep learning [12]. Despite these advances, a key challenge remains: how to integrate learning-based components within a set-membership estimation framework while preserving guaranteed bounds on the state estimate. In particular, the interaction between nonlinear neural models and set-based uncertainty propagation is not straightforward, and often leads to either loss of guarantees or overly conservative approximations. In this paper, a hybrid framework for state estimation is proposed, that combines a linear Model-based prediction inpired by the Kalman filtering, a neural correction mechanism, and constrained zonotope-based uncertainty propagation. The proposed approach relies on a structured decomposition of the estimation problem: the state estimate is represented as the sum of a learned center, obtained through a neural correction of the Model-based prediction, and a bounded uncertainty set represented by a constrained zonotope. The neural network is trained to capture residual dynamics and compensate for model inaccuracies. It allows to bypass the unrealistic hypothesis induced in the Kalman filtering approach, while uncertainty is propagated separately through set-based techniques. More specifically, the contributions of this work are threefold, a hybrid estimation architecture combining a physical prediction model with a neural correction term based on recurrent neural networks, enabling the learning of residual dynamics while preserving model structure, a set-membership uncertainty propagation framework based on constrained zonotopes, allowing the computation of guaranteed bounds on the estimation error under bounded disturbances, a structured integration of learning and set-based estimation, where the neural component is used to estimate the center of the uncertainty set, while the

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associated uncertainty is propagated using a Jacobian-based linearization approach. The effectiveness of the proposed method is demonstrated on a benchmark system, highlighting its ability to improve estimation accuracy compared to purely model-based approaches while maintaining reliable uncertainty bounds. This paper is organized as follows. Section II formulates the state estimation problem under uncertainty. Section III introduces the necessary background on constrained zonotopes and neural network architectures. Section IV presents the proposed hybrid estimation framework. Section V provides simulation results, and Section VI concludes the paper and outlines future research directions.

II. Problem formulation

The fundamental objective of filtering in dynamical systems is state estimation, which involves determining the state vector $x_k \in \mathbb{R}^n$ at time instant k that completely characterizes the system's behavior. In this work, we consider linear time-invariant discrete-time systems described by :

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k, \\ y_k = Cx_k + v_k, \end{cases} \tag{1}$$

where $u_k \in \mathbb{R}^{n_u}$ is the input vector, $y_k \in \mathbb{R}^{n_y}$ is the measurements vector, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_u}$ and $C \in \mathbb{R}^{n_y\times n}$ are the state transition, input and output matrices, respectively and w_k and v_k represent process and measurement uncertainties, respectively. State estimation in such systems critically depends on how uncertainties are modeled. A classical approach relies on probabilistic assumptions, where the noise terms are modeled as random variables with known distributions. In particular, [1] which assumes that $w_k \sim \mathcal{N}(0, Q)$ and $v_k \sim \mathcal{N}(0, R)$, where Q and R are the covariance matrices of w_k and v_k respectively, within a linear Gaussian system. The initial state, and the noise vectors at each step $\{x_0, w_1, \ldots, w_k, v_1, \ldots, v_k\}$ are all assumed to be mutually independent. The Kalman filter operates through recursive prediction and update steps, for $k \in \mathbb{N}^*$:

- Prediction

$$\begin{aligned} \hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1} + Bu_{k-1}, \\ P_{k|k-1} &= AP_{k-1|k-1}A^T + Q \end{aligned} \tag{2}$$

- Correction

$$\begin{aligned} K_k &= P_{k|k-1}C^T(CP_{k|k-1}C^T + R)^{-1}, \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}) \end{aligned} \tag{3}$$

While a set-membership approaches represent uncertainties as unknown but bounded quantities, without relying on statistical assumptions. In this framework, the noise terms are assumed to belong to known bounded sets, and the objective is to compute a set of all possible states consistent with the model and the measurements [13], [3], [4]. Among the different set representations, zonotopes have emerged as a particularly efficient choice due to their favorable computational properties and their ability to propagate uncertainty through linear systems. However, both probabilistic and set-membership approaches present inherent limitations when used independently. More recently, data-driven approaches based on machine learning have been introduced to address model inaccuracies by learning system behavior directly from data [8]. In particular, neural networks are capable of capturing complex nonlinear relationships and temporal dependencies. Nevertheless, these approaches require large amounts of data, may introduce additional computational latency, and do not provide formal guarantees on the estimation error. To overcome the limitations of individual methodologies, hybrid estimation strategies have been proposed, combining complementary advantages of model-based, set-membership, and data-driven approaches. For instance, probabilistic and set-based methods have been combined to balance statistical efficiency and robustness [11], while other works integrate learning-based models into classical filtering frameworks to improve adaptability [9]. In this work, we adopt a hybrid perspective and propose a framework that combines probabilistic filtering, set-membership uncertainty representation, and data-driven correction mechanisms. The objective is to leverage the strengths of each paradigm: fast recursive estimation from probabilistic filtering, guaranteed uncertainty representation from set-membership methods, and adaptability to model mismatch from learning-based approaches. This integrated framework aims to provide a balanced compromise between estimation accuracy, robustness, and computational efficiency, enabling reliable state estimation under a wide range of uncertainty conditions.

III. Definitions and notations

A. Constrained zonotope

The definition of constrained zonotopes is given in [5].

Definition 1: A set $\mathcal{Z}^C \subset \mathbb{R}^n$ is a constrained zonotope if there exist matrices $G \in \mathbb{R}^{n\times n_g}$, $A^C \in \mathbb{R}^{n_c\times n_g}$ and some vectors $c \in \mathbb{R}^n$ and $b^C \in \mathbb{R}^{n_c}$ such that:

$$\mathcal{Z}^C = \left\{G\xi + c \mid \|\xi\|_\infty \leq 1,\ A^C\xi = b^C\right\}, \tag{4}$$

where ξ denotes the unit hypercube in $\mathbb{R}^{n_g}$, and $\|\xi\|_\infty$ is the infinity norm, let $\xi = [\xi_1, \ldots, \xi_{n_g}]^\top$, then $\|\xi\|_\infty = max(|\xi_1|, \ldots, |\xi_{n_g}|)$.
The shorthand notation $\mathcal{Z}^C = \langle G, c, A^C, b^C\rangle$ will be used in that follows. Constrained zonotopes provide a flexible representation of sets while enabling efficient implementation of linear operations such as affine transformation, Minkowski sum, and intersection. The constrained zonotope representation admits several useful properties that enable efficient set operations. The following results are adapted from [5].

Property 1 (Linear transformation): Given a constrained zonotope $\mathcal{Z}^C = \langle G, c, A^C, b^C\rangle \subset \mathbb{R}^n$ and a matrix $R \in \mathbb{R}^{n_R\times n}$, the linear transformation is given by:

$$R\mathcal{Z}^C = \langle RG, Rc, A^C, b^C\rangle. \tag{5}$$

Property 2 (Minkowski sum): Given two constrained zonotopes $\mathcal{Z} = \langle G_z, c_z, A_z, b_z\rangle$ and $\mathcal{W} = \langle G_w, c_w, A_w, b_w\rangle$ in $\mathbb{R}^n$, their Minkowski sum is:

$$\mathcal{Z} \oplus \mathcal{W} = \left\langle \begin{bmatrix} G_z & G_w \end{bmatrix},\ c_z + c_w,\ \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix},\ \begin{bmatrix} b_z \\ b_w \end{bmatrix} \right\rangle \tag{6}$$

Property 3 (Linear intersection): Given a constrained zonotope $\mathcal{Z} = \langle G_z, c_z, A_z, b_z \rangle \subset \mathbb{R}^n$, another constrained zonotope $\mathcal{Y} = \langle G_y, c_y, A_y, b_y \rangle \subset \mathbb{R}^k$, and a matrix $R \in \mathbb{R}^{k \times n}$, the intersection under the linear map R is:

$$\mathcal{Z} \cap_R \mathcal{Y} = \left\langle \begin{bmatrix} G_z & 0 \end{bmatrix}, \ c_z, \ \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix} \right\rangle \tag{7}$$

When both zonotopes share the same space $\mathbb{R}^n$ (i.e., $k = n$ and $R = I_n$).

These properties are fundamental for set-based estimation algorithms as they allow efficient propagation of uncertainty through linear system dynamics and measurement models while maintaining the constrained zonotope representation.

B. Gated Recurrent Unit and Fully Connected Layer

Definition 2: The GRU, introduced by [6], is a recurrent neural network (RNN) architecture designed to address the limitations of standard RNNs, particularly the vanishing gradient problem and the difficulty in learning long-term dependencies. It offers a structurally simpler and often computationally more efficient alternative to LSTM units [2], [14], [7].

The fundamental mechanism of the GRU relies on two gates that regulate information flow (the input gate and the reset gate). Formally, at time step k, given the input y_k and the previous hidden state h_{k-1}, the computations proceed as follows:

- The **update gate** controls how much of the previous memory is preserved by compting z_k a vector of values between 0 and 1 via a sigmoid function σ, it aims to determines the proportion in which the previous state h_{k-1} is preserved:

$$z_k = \sigma(W_z[h_{k-1}, y_k] + b_z) \tag{8}$$

  Where W_z and b_z are respectively the weight matrix and the bias of the update gate.
- The **reset gate** r_k is computed similarly:

$$r_k = \sigma(W_r[h_{k-1}, y_k] + b_r) \tag{9}$$

  It indicates how much of the past state is used to compute the new candidate state. A low value "resets" or forgets the past, letting the network focus on new input.
- The **candidate hidden state** $\tilde{h}_k$ is generated by combining the current input with a filtered version of the previous state via the reset gate r_k:

$$\tilde{h}_k = \tanh(W_h[r_k \odot h_{k-1}, y_k]) \tag{10}$$

  where $\odot$ denote the Hadamard product.
- The final **hidden state** h_k results from an interpolation between the previous state h_{k-1} and the candidate state $\tilde{h}_k$, weighted by the update gate z_k:

$$h_k = (1 - z_k) \odot h_{k-1} + z_k \odot \tilde{h}_k \tag{11}$$

  This equation is central to the GRU's ability to preserve certain information over the long term (when z_k is close to 0) or to update it significantly (when z_k is close to 1).

Here, h_{k-1}, z_k, r_k and $\tilde{h}_k$ are the local variables of the GRU unit. They correspond to intermediate computations within a single GRU unit, used to update the hidden state from h_{k-1} to h_k.

Definition 3: As described by [14], in the Fully Connected (FC) layer, each neuron is connected to every neuron in the preceding layer, following the structure of a standard multi-layer perceptron.

The integration of fully connected layers after recurrent architectures like GRU enables the network to combine temporal dependencies learned from sequential data with the discriminative power of multi-layer perceptrons, ultimately allowing the model to make final predictions based on the accumulated contextual information.

IV. Neural-Model-Based Estimation with Constrained Zonotope Based Uncertainty Propagation

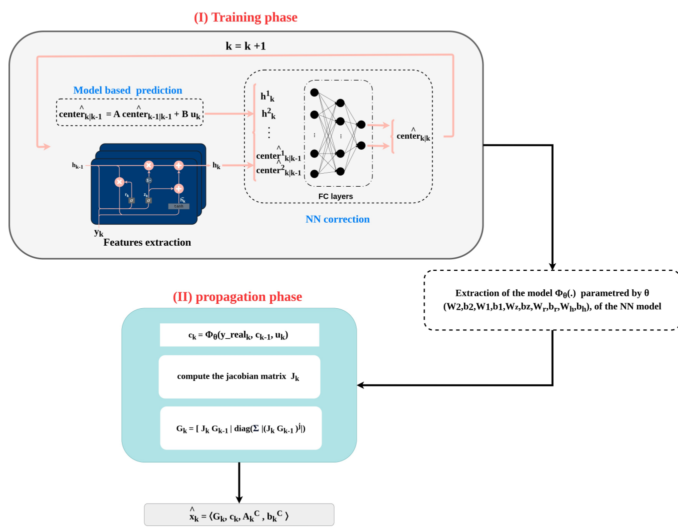

Fig. 1: Overview of the proposed hybrid approach.

The complete methodology is summarized in Figure 1 and can be described as follows: it consists of two main phases: (1) a training phase where the neural network Φ_θ is learned, and (2) a propagation phase where uncertainty is propagated through constrained zonotopes. The separation between center estimation and uncertainty propagation is a key feature of the approach. During the training phase, the neural network model Φ_θ (parametrized by θ) is learned. The architecture combines GRU layers for temporal dependencies and fully connected (FC) layers for state prediction. The Model-based prediction provides an initial estimate, which is then corrected by the neural network. Once trained, the model is used to propagate uncertainty represented as constrained zonotopes. At each time step k, the center of the zonotope is propagated through the neural network and the uncertainty is propagated through linearization, with an additional term accounting for linearization error. The separation of the two phases allows the method to leverage the adaptability of learning-based models while maintaining a structured representation of uncertainty.

A. Propagation and correction of the centers

The first stage of the proposed framework focuses on the estimation of the center of the uncertainty set. This is achieved by combining Model-based prediction step based on the known physical model with a neural correction mechanism designed to compensate for model inaccuracies using real data training. It aims to predict the centers of a constrained zonotopes $\hat{center}_k$. The baseline prediction is given by the linear model:

$$\hat{center}_{k|k-1} = A\hat{center}_{k-1|k-1} + Bu_k \tag{12}$$

Equation (12) corresponds to the nominal model-based prediction, this differs from classical Kalman filtering: we do not propagate covariance matrices or compute optimal Kalman gains. Instead, uncertainty is represented and propagated through constrained zonotopes as a complement to the neural correction mechanism. which provides a baseline estimate of $\hat{center}_{k|k-1}$, the prior center of the constrained zonotope. However, due the unrealistic hypothesis on the noise, the model mismatch and unmodeled dynamics, this estimate may be biased. Simultaneously, temporal information is extracted from the noisy measurements y_k using a GRU network:

$$h_k = GRU_\rho(y_k) \tag{13}$$

where $h_k \in \mathbb{R}^{n_h}$ is the hidden state from the last GRU layer at time step k, $GRU(.)$ denotes the stacked GRU network, and ρ represents the learnable parameters (weight matrices and biases) of the GRU.
The neural correction pathway is then fused with the physical prediction via concatenation:

$$v_k = \begin{bmatrix} h_k \\ \hat{center}_{k|k-1} \end{bmatrix} \tag{14}$$

where v_k combines the temporal features learned by the GRU with the physically predicted state. Finally, the posterior center estimate is computed through a shallow feedforward network with a hyperbolic tangent activation:

$$\hat{center}_{k|k} = W_2\Big(\tanh\big(W_1 v_k + b_1\big)\Big) + b_2 \tag{15}$$

where W_1, b_1 and W_2, b_2 are the weight matrices and biases of the two fully connected layers, and tanh provides a bounded nonlinear transformation. This network learns to compensate for model inaccuracies and unmodeled disturbances, producing a corrected state estimate that blends physical prior knowledge with data-driven temporal features. The neural component in the overall framework aims to account for modeling inaccuracies, unmodeled dynamics, and measurement noise that cannot be captured by the physical model alone. The nominal representation of the system matrices A and B is reliable; however, nonlinearities, parameter variations, and external disturbances inevitably affect real-world systems. Thus, the neural network should correct the physical model rather than substitute for it. The hybrid formulation offers several advantages, It preserves the interpretability and stability of the physical model, it reduces data requirements compared to fully data-driven models, it improves generalization by constraining learning to residual dynamics.

The optimization objective is formulated as a mean squared error (MSE) minimization problem between the neural-corrected state estimates and the true system states:

$$\mathcal{L}(\theta) = \frac{1}{K}\sum_{k=1}^{K} ||\hat{center}_k - center_k^{\text{Targets}}||_2^2 \tag{16}$$

where θ denotes trainable parameters from 13 to 15, and $center_k^{\text{Targets}}$ is the ground truth centers of the constrained zonotopes at time step k, and K is the total length of the trajectory.

B. Constraint zonotope Uncertainty Propagation

The second stage of the proposed framework handles the propagation of estimation uncertainty. Although the estimate's center is determined via the nonlinear mapping $\Phi_\theta(.)$ resulting from the neural network described from the equation 12 to the equation 15) and parameterized by θ which represent the weights matrix and biases for the GRU and FC layers for the full learning model, as illustrated in Figure 1 bloc (I). The associated uncertainty is propagated in a computationally tractable manner using constrained zonotopes. The state estimation is given by :

$$\hat{x}_k = \Phi_\theta(y_k, \hat{center}_{k-1|k-1}, u_k) + \xi_k, \tag{17}$$

where $\Phi_\theta(.)$ is a nonlinear function and ξ_k represents the estimation error of the centers. The error in included in a constrained zonotope $\mathcal{E}_k$, so 17 becomes 18 as :

$$\hat{x}_k = \Phi_\theta(y_k, \hat{center}_{k-1|k-1}, u_k) \oplus \mathcal{E}_k \tag{18}$$

where $\xi \in \mathcal{E}_k$, here $\hat{x}_k$ is a constrained zonotope as :

$$\hat{x}_k = \langle G_k, \hat{center}_{k|k}, A_k^C, b_k^C \rangle, \tag{19}$$

where G_k, A_k^C, b_k^C are repectively the generator matrix, the constrained matrix and the contrained vectors of the contraines zonotope $\hat{x}_k$

Since the neural mapping Φ_θ direct propagation of the uncertainty set is intractable. Therefore, a local linearization around the center $\hat{center}_{k|k}$ is employed to approximate the effect of the nonlinear mapping on the uncertainty:

$$\begin{aligned} \Phi_\theta(y_k, center_k, u_k) = \Phi_\theta(y_k, \hat{center}_{k|k}, u_k) \\ + J_k(center_k - \hat{center}_{k|k}) \\ + \mathcal{O}(\|center_k - \hat{center}_{k|k}\|^2) \end{aligned} \tag{20}$$

Where the term err_k captures the approximation error induced by the linearization. This term is bounded and incorporated into the uncertainty set. J_k is the Jacobian matrix of the network with respect to the state:

$$J_k = \frac{\partial \Phi_\theta}{\partial center}\bigg|_{(y_k, \hat{center}_{k|k}, u_k)} \tag{21}$$

it is evaluated at $\hat{center}_{k|k}$, and err_k is bounded by a constrained zonotope $err_k \in \mathcal{E}_k$.
The centers of the constrained zonotopes is propagated using Φ_θ:

$$\hat{center}_{k|k} = \Phi_\theta(y_k, \hat{center}_{k-1|k-1}, u_k) \tag{22}$$

Based on this linear approximation, the generators of the constrained zonotope are propagated through the Jacobian of the neural model.

$$G_k = [J_k G_{k-1} | G_k^{\text{err}}] \tag{23}$$

We concatenated G_k^{err} an encoded as an additional set of generators:

$$G_k^{\text{err}} = \text{diag}(\sum_j |(J_k G_{k-1})^j|) \tag{24}$$

where $(J_k G_{k-1})^j$ denotes the j-th column of the matrix. This bound represents the maximum possible deviation of the constrained zonotope.

$$x_k = \langle G_k, \hat{center}_{k|k}, A_k^C, b_k^C \rangle \tag{25}$$

This construction provides a conservative bound on the effect of the nonlinear transformation, ensuring that the resulting zonotope encloses all possible propagated states under the linearization assumption.

Proof: The neural network Φ_θ is Lipschitz continuous with constant $L = \max_k \|J_k\|_2$. Expanding around the estimated center with the equation (20). The uncertainty from the previous step lies in G_{k-1}, so the first-order term $J_k(c_k - \hat{c}_{k|k-1})$ is exactly captured by $J_k G_{k-1}$. The higher-order terms and linearization errors are bounded by the Lipschitz continuity and absorbed into G_k^{err} equation (24). By zonotope properties, all error sources are contained in $\langle 0, G_k^{\text{err}} \rangle$, so $x_k = \hat{x}_k^{\text{pred}} + e_k$ where $e_k \in \langle 0, G_k^{\text{err}} \rangle$. Therefore $x_k \in \langle \hat{c}_{k|k}, [J_k G_{k-1} | G_k^{\text{err}}] \rangle = \hat{x}_k$. □ ■

V. Simulation Results

The performance of the proposed learning-enhanced set-membership estimation framework is evaluated on a vehicle lateral dynamics system. The nonlinear bicycle model is linearized around the nominal operating point to capture the essential lateral behavior while maintaining computational tractability. The continuous-time state-space representation is defined as:

$$\begin{bmatrix} \dot{\beta}(t) \\ \ddot{\psi}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{-C_f - C_r}{mv} & 1 + \dfrac{\mu(-l_r C_r - l_f C_f)}{mv^2} \\ \dfrac{-l_r C_r - l_f C_f}{I_z} & \dfrac{-l_f^2 C_f - l_r^2 C_r)}{I_z v} \end{bmatrix}}_{A} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} \dfrac{C_f}{mv} & 0 & 0 & 0 \\ \dfrac{l_f C_f}{I_z} & \dfrac{1}{I_z} & \dfrac{S_r R t_r}{2I_z} & -\dfrac{S_r R t_r}{2I_z} \end{bmatrix}}_{B} \begin{bmatrix} \delta \\ M_{dz} \\ T_{bl} \\ T_{br} \end{bmatrix}. \tag{26}$$

where the state vector $\mathbf{x}(t) = [\beta(t)\, \dot{\psi}(t)]^T$ comprises the sideslip angle and yaw rate, and the input vector includes steering angle δ, yaw moment disturbance M_{dz}, and braking torques T_{bl}, T_{br}. The parameters C_f, C_r (tire cornering stiffnesses), l_f, l_r (distances from center of gravity to front/rear axles), I_z (yaw inertia), m (vehicle mass), and v (velocity) are vehicle parameters determined through the linearization procedure. The output vector comprises both state derivatives and state variables $y(t) = \begin{bmatrix} \dot{\beta}(t) & \beta(t) & \ddot{\psi}(t) & \dot{\psi}(t) \end{bmatrix}^T$. The measurement equation $y(t) = C\mathbf{x}(t)$ maps the internal state space to the measurement space. The output matrix C is constructed to extract both state and state-derivative information: $C = \begin{bmatrix} A_{1,:}^\top & [1\ 0]^\top & A_{2,:}^\top & [0\ 1]^\top \end{bmatrix}^\top$. This structure ensures full observability: the first and third rows encode the dynamics (from matrix A), allowing reconstruction of state derivatives, while the second and fourth rows directly measure the states. For implementation, the continuous-time model is discretized with sampling period $T_s = 0.05\,\text{s}$ using zero-order hold discretization. The resulting discrete-time state-space model is:

$$\begin{cases} \mathbf{x}_{k+1} = A_d \mathbf{x}_k + B_d \mathbf{u}_k \\ y_k = C_d \mathbf{x}_k + D_d \mathbf{u}_k \end{cases} \tag{27}$$

where the discrete-time system matrices are:

$$A = \begin{bmatrix} -0.52 & -0.013 \\ -3.72 & -0.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 & 0 & 0 & 0 \\ 13.03 & 0.0004 & 1.24 & -0.01 \end{bmatrix}.$$

The output matrix C retains the structure of the continuous case, and observability properties are preserved under the discretization.

The simulation runs over $N = 864$ time steps. The vehicle travels at a known and constant speed of $v = 15\,\text{m/s}$ ($54\,\text{km/h}$) on a dry road ($\mu = 1$), starting from the initial state $x_0 = (0,\ 0)^T$. Further details on the linearization and the full parameter set are provided in [15]. The training dataset is generated by simulating the dynamical system over multiple scenarios, with the addition perturbations and uncertainties. This dataset is used to form a test set with added uncertainties by adding zero-mean Gaussian measurement noise $\mathcal{N}(0,\ \sigma^2)$, with $\sigma = 0.01$, to the nominal output measurements $y(t)$. The dataset is split into 80% training, 10% validation, and 10% test sets. The clean state trajectories $x(t)$ serve as regression targets, and quantitative metrics are computed to evaluate the model performance. For the set-membership estimation, a constrained zonotope is propagated over the full trajectory. The initial zonotope is defined by center $c_0 = (0,\ 0)^T$ and generator matrix $G_0 = 0.05\, I_2$. At each time step, the zonotope center is updated via the learned estimator, and the generator matrix is propagated through the Jacobian of the network. Table 1 summarizes the results of the performance evaluation.

Metric	Value
MSE	0.000123
RMSE	0.011095
MAE	0.008456

TABLE I: Performance metrics on the test set

The metrics provided (MSE, RMSE, MAE) suggest a high level of accuracy in the estimation of the state centers. Here, the RMSE is the square root of MSE, and it gives the error in the same units as the predicted variable. On the other hand, MAE gives the average absolute error. However, these metrics alone do not capture the reliability of the uncertainty representation, which is a key aspect of the proposed method. Figure 2 presents a detailed comparison between the true and predicted trajectories for a test scenario, illustrating the evolution of both the first and second coordinates of the $\hat{center}_k$. The results show that the proposed model accurately captures the system dynamics, with the predicted centers closely following the true values. Left part

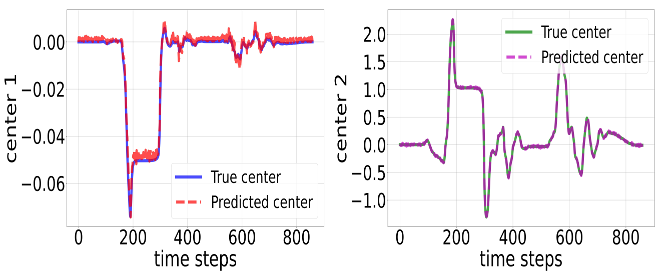

Fig. 2: Predicted centers of states: Left center 1, Right center 2.

of Figure 3 presents the estimated state together with the uncertainty bounds derived from the constrained zonotope representation. The true state remains consistently enclosed within the estimated bounds, indicating that the proposed uncertainty propagation provides a reliable envelope of the system state. The relatively tight bounds derived from the constrained zonotope propagation described in Section IV-B proves that the method effectively limits conservatism while maintaining robustness. Right part of Figure 3 further illustrates the behavior of the uncertainty bounds under more dynamic variations (wheels dynamics). The proposed approach accurately captures the dynamic variations, including the periods of higher dynamics. The experimental validation

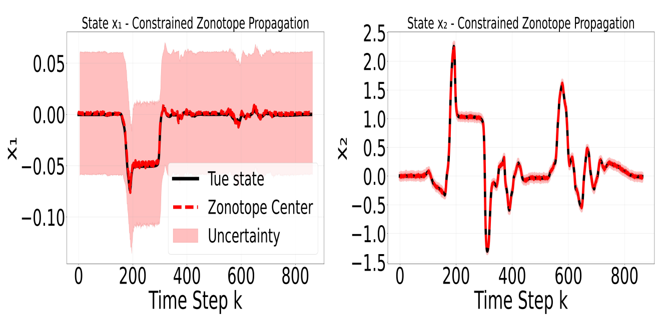

Fig. 3: True state vs estimated center with uncertainty bounds (shaded region). Left (x_1) Right (x_2).

demonstrates several key strengths of the proposed learning-enhanced set-membership state estimation framework. The results presented in Figure 3 show that the estimated centers closely follow the true states throughout the trajectory. This accuracy is quantitatively confirmed by the low error metrics reported in Table I, with an MSE of 0.000123 and an MAE of 0.008456. The neural correction component effectively compensates for unmodeled dynamics and noise assumptions while preserving the physical model's structure, reducing the conservatism while the zonotopic propagation maintain the robustness of the estimation.

VI. Conclusion and Future Works

In this paper, a learning-enhanced set-membership state estimation framework is proposed, combining a model-based prediction, a neural correction mechanism, and constrained zonotope-based uncertainty propagation. By separating center estimation from uncertainty representation, the approach leverages the adaptability of learning-based models while maintaining a structured and interpretable description of uncertainty. The results show that the proposed method improves estimation accuracy while providing reliable uncertainty bounds that adapt to the system dynamics. This highlights the relevance of combining data-driven correction with set-membership techniques for robust state estimation under complex and partially unknown disturbances. Some limitations remain, particularly regarding the use of local linearization for uncertainty propagation and the evaluation on relatively simple systems. Future work will focus on strengthening the theoretical guarantees of the approach and extending its validation to more complex and high-dimensional scenarios.

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