Observer Design for Allen-Cahn Reaction-Diffusion Systems

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***Abstract*—This paper proposes an observer design methodology for Allen–Cahn reaction–diffusion systems, a class of nonlinear parabolic partial differential equations (PDEs). The observer is constructed from a finite-dimensional ordinary differential equation (ODE) approximation of the original PDE, obtained via the method of lines through spatial discretization while retaining continuous-time dynamics. Using Lyapunov-based analysis, it is shown that the proposed observer achieves convergence of the estimation error over the entire solution domain, despite the presence of the sign-changing nonlinear reaction term. The required number of measurement points and their spatial placement, necessary for guaranteed estimation performance, can be determined using the standard linear observability condition. Numerical simulations validate the proposed approach and demonstrate accurate state reconstruction under process and measurement noise.**

***Index Terms*—Nonlinear Observers, Reaction-Diffusion Systems, Partial Differential Equations, Lyapunov analysis**

I. Introduction

The Allen-Cahn PDE is a reaction-diffusion equation that plays an important role in the modeling of dynamic phase transitions with various applications in different fields of science such as mathematical biology [1] or material science [2]. This equation provides a mathematical framework for studying the dynamics of interfaces, such as the evolution of boundaries between different phases or states of matter.

Many analytical and numerical schemes were developed to find or approximate the solution of Allen-Cahn PDE. Semi-analytical Fourier spectral methods of first and second order for solving the Allen–Cahn equation were proposed in [3]. In [4], the authors study the analytical, semi-analytical, and numerical solutions of the Allen-Cahn equation that describes the dynamics for the phase separation in some ternary alloys. A spatially fourth-order hybrid numerical scheme was given to approximate the solution of the 2D and 3D Allen-Cahn problem in [5]. An accurate approximate solution was also given in [6] using the explicit finite difference method.

Due to the wide range of practical applicability of such PDEs, the observer design for these models is a critical area of control theory and applied mathematics [7]. The PDE observers aim to estimate the states of distributed systems which evolve both in space and time. In practical scenarios, full-state measurement in such processes is often infeasible due to limitations in sensor placement or measurement capabilities. Observers bridge this gap by using limited sensor data to reconstruct the PDE solution in real-time. Moreover, the observers can also deal with bounded disturbances and sensor noises.

Numerous state observer design methods have been developed for various dynamic processes modeled by ODEs [8]. Some of these state estimation techniques for ODE systems can also be adapted for PDE systems. For instance, [9] introduces an ODE model and co-designs a state observer tailored for PDEs that effectively captures the dynamics of buoyancy-driven turbulent flows. The challenge of sensor placement to efficiently reconstruct infinite-dimensional solutions of distributed process systems was addressed in [10]. A comprehensive review of state observer design methods for PDEs is provided in [7].

Most observer design methods for PDEs assume that the nonlinear source terms in the models satisfy the Lipschitz condition. Indeed, the survey paper [7] explicitly identifies the Lipschitz property as a fundamental assumption underlying all the discussed observer design methods. However, the Lipschitz condition could impose Lipschitz constant-dependent restrictions on the observer gains. This fact hinders the applicability of such efficient estimation methods as the linear quadratic estimators [11]. The approach taken in this work combines a common, unconstrained observer design method, and a high gain observer design to ensure the asymptotic stability of the estimation error.

A common strategy in the literature is to formulate PDE observer design via linear matrix inequality (LMI) conditions; see, for example, [12] and [13]. In contrast, the observability condition for the estimator design is more straightforward in this work: the approximate ODE must meet the well-known matrix rank condition applied in linear time-invariant systems [11].

The highlights of this study can be summarized as:

- We showed that the approximate ODE model of the Allen-Cahn equation, which is developed based on the spatial discretization of the original PDE, can efficiently be applied to state observer design.
- We developed a nonlinear state observer for Allen-Cahn systems, which applies a switching observer gain. Using

Lyapunov methods, we showed that the proposed observer guarantees the convergence of the estimated states to the real states of the Allen-Cahn system in the entire state space.

The rest of this paper proceeds as follows. A suitable ODE model for the design of the PDE observer is presented in section II. Section III introduces the proposed observer design method. The applicability of the proposed PDE observer is shown in section IV. Simulation measurements are also presented here. Finally, Section V summarizes the conclusions of this study.

Notations:

- $\mathbb{R}_+$, $\mathbb{R}_+^*$ – sets of positive and strictly positive real numbers. $\mathbb{N}^*$ – set of strictly positive natural numbers
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, $\mathbf{1} \in \mathbb{R}^n$ is the all-ones vector, $\mathbf{0} \in \mathbb{R}^n$ is the all-zeros vector.
- Let $\boldsymbol{\beta} = (\beta_i)_{1\le i\le n} \in \mathbb{R}^n$. The Hadamard (entry-wise) product of two vectors is $\boldsymbol{\beta_1} \circ \boldsymbol{\beta_2} = (\beta_{1i}\beta_{2i})_{1\le i\le n}$, and the entry-wise mth power of a vector $\boldsymbol{\beta}^m = (\beta_i^m)_{1\le i\le n}$, $m \in \mathbb{N}^*$.
- Let $A^\top$ the transpose of a matrix $A \in \mathbb{R}^{n\times n}$. Let $\mathrm{sym}(A) = (A + A^\top)/2$.
- The operator $\otimes$ denotes the Kronecker product

II. ODE approximation of Allen-Cahn PDE

Consider the initial value problem [14]

$$\frac{\partial u}{\partial t} = q\nabla^2_{\mathbf{x}} u + \underbrace{u(\gamma - u)(u-\theta)}_{r(u)} + f,\ \forall t > 0, \mathbf{x} \in \Omega,$$

$$u\Big|_{\partial\Omega} = 0,\ \forall t > 0,\ \text{and}\ u\Big|_{t=0} = u_0(\mathbf{x}),\ \forall \mathbf{x} \in \Omega, \quad (1)$$

where $\Omega \subseteq \mathbb{R}^n$ is an open region with $\partial\Omega$ boundary, $u :- u(t, \mathbf{x}) \in \mathbb{R}_+$ is the solution of the equation (state of the system); $\mathbf{x} \in \Omega$ is a spatial point, t is the time coordinate, $q > 0$ is the diffusion coefficient, $r(u) \in \mathbb{R}_+$ is the reaction term, and $\gamma \ge \theta > 0$ are the reaction term parameters. $f :- f(\mathbf{x}, t) \in [0, \gamma)$ is an additional Lipschitz continuous source term, which is vanishing (i.e. $f \to 0$ as $t \to \infty$, $\forall x \in \Omega$). This system is called the Allen-Cahn equation with Dirichlet-type boundary condition, and $u_0(\mathbf{x}) : \Omega \to [0, 1]$ is the normalized initial function.

Remark 1: The term $r(u) + f$ is always non-negative. According to the theorem on invariant regions for reaction-diffusion systems, the positivity of the dynamics is guaranteed, i.e., $u(x,t) \ge 0$ for all $(\mathbf{x}, t)$ given non-negative initial conditions [15]. The assumption $f(\mathbf{x}, t) \in [0, \gamma)$ is essential here: it ensures that the invariant-region argument also yields the upper bound $u(\mathbf{x}, t) \le \gamma$, confining the solution to $[0, \gamma]$.

Remark 2: Consider an ODE in which the state rate is equal to the reaction term in (1), i.e. $\dot{u} = r(u),\ u(0) = u_0 \ge 0$. It is a bi-stable equation with two stable equilibria ($u = 0$ and $u = \gamma$) and one unstable ($u = \theta$) equilibrium state. If the initial state is in the interval $(0, \theta)$, the ODE state converges to 0. If $u_0 \in (\theta, \gamma)$ the state converges to γ. In mathematical ecology, such models can be applied to describe the Allee effect [16]. In this case, the state of the system can represent the population size in a habitat, the parameter γ is the carrying capacity of the habitat, and the parameter θ is the critical population size. If the initial population is smaller than θ, the population dies out. Otherwise, the population size converges to γ. Similar effects can be expected in the case of the PDE (1). Especially for small q values, the parameter θ also plays a decisive role in the evolution of the PDE solution [14].

There are a multitude of approximation schemes for the solution of the Allen-Cahn PDE, such as Euler [17] or Galerkin methods [18]. By applying only spatial discretization, the numerical stability problems, which may arise during the simultaneous time- and space discretization, see e.g. [19], can be avoided. The spatial Galerkin approximation [20] and the method of lines [21] are suitable since they only discretize the spatial domain. In this paper, we opted for the method of lines in which the continuous Laplacian operator is approximated using a two-sided difference. At a spatial grid (lattice) point, indexed by $\mathcal{G} = (i_1, \dots i_j, \dots i_n)$, the discretized Laplacian is:

$$\nabla^2_{\mathbf{x}} u \approx D^2_{\mathbf{x}} u = \sum_{j=1}^{n} \frac{u_{\mathcal{N}j-} - 2u_{\mathcal{G}} + u_{\mathcal{N}j+}}{\Delta x^2} \quad (2)$$

where $u_{\mathcal{G}}$ represents the values of u at the grid point indexed by $\mathcal{G}$, the subindex $\mathcal{N}j+$ means moving forward in the jth coordinate, $\mathcal{N}j-$ means moving backward in the jth coordinate. $\Delta x > 0$ is the grid resolution, applied to spatial discretization.

This decomposition results in a large, interconnected state space with sparse state matrix: $A = (q/\Delta x^2)(a_{ij})_{1\le i,j\le m^n} \in \mathbb{R}^{m^n\times m^n}$ where:

$$a_{ij} = \begin{cases} -2n, & \text{if } i = j, \\ 1, & \text{if nodes } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Here $m \in \mathbb{N}^*$ is the number of nodes, or the number of sections along one coordinate of the uniformly divided space. The case $n = 2$ for the matrix A is illustrated in Eq. (23).

Thus, with Dirichlet boundary conditions as specified in (1), and by applying the Laplacian matrix A defined in (3), we arrive at the approximating system

$$\dot{\boldsymbol{\beta}} = A\boldsymbol{\beta} + \boldsymbol{\beta} \circ (\gamma\mathbf{1} - \boldsymbol{\beta}) \circ (\boldsymbol{\beta} - \theta\mathbf{1}) + \mathbf{f} + \mathbf{w},\ \forall t > 0 \quad (4)$$

where $\boldsymbol{\beta} = \boldsymbol{\beta}(t) \in \mathbb{R}_+^{m^n}$ represents the approximate PDE solution vector in the lattice points. Let the initial condition $\boldsymbol{\beta}_k(0) = u(\mathbf{x}_k, 0)$, where $\mathbf{x}_k$ is the spatial lattice $k = 0, 1, \dots, m^n$. The source vector $\mathbf{f} \in R_+^{m^n}$ contains the values of f in the lattice points. $\mathbf{w} \in \mathbb{R}^{m^n}$ denotes a bounded unknown uncertainty term representing the approximation error.

Example: Consider the case when $n = 1$, i.e., the spatial domain is one-dimensional ($\Omega \subseteq \mathbb{R}$). In this case, the state dynamics in the ith spatial compartment ($i = 1 \dots m$) reads as:

$$\dot{\beta}_i = q\beta_{i-1} - 2q\beta_i + q\beta_{i+1} + \underbrace{\beta_i(\gamma - \beta_i)(\beta_i - \theta)}_{r_i(\beta_i)} + f_i + w_i \quad (5)$$

We examine the approximation property of the discrete Laplacian approximation for the diffusion term in the equation.

Proposition: The finite difference approximator $D^2_{\mathbf{x}}u$ of $\nabla^2_{\mathbf{x}}u$ at an $\mathbf{x}_i$ point has a truncation error of $\mathcal{O}(\Delta x^2)$.

Proof: Starting from the finite difference relation (2), we expand u around the point $\mathcal{G}$ along each coordinate direction using the Taylor series

$$u_{\mathcal{N}j+} = u_{\mathcal{G}} \\ -\frac{\partial u}{\partial x_k}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x_k^2}\Delta x^2 - \frac{1}{3!}\frac{\partial^3 u}{\partial x_k^3}\Delta x^3 + \mathcal{O}(\Delta x^4). \quad (6)$$

Similarly, we have

$$u_{\mathcal{N}j-} = u_{\mathcal{G}} \\ +\frac{\partial u}{\partial x_k}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x_k^2}\Delta x^2 + \frac{1}{3!}\frac{\partial^3 u}{\partial x_k^3}\Delta x^3 + \mathcal{O}(\Delta x^4). \quad (7)$$

Substituting (6) and (7) into (2), every odd term is canceled, and it yields

$$D^2_{\mathbf{x}}u = \sum_{k=1}^{n}\frac{1}{\Delta x^2}\Big(\frac{\partial^2 u}{\partial x_k^2}\Delta x^2 + \mathcal{O}(\Delta x^4)\Big) \\ = \nabla^2_{\mathbf{x}}u + \mathcal{O}(\Delta x^2). \quad (8)$$

Thus in each $\mathbf{x}_k$ point for $k = 0, 1, \dots, m^n$, as $\Delta x \to 0$, the discrete Laplacian converges to the continuous Laplacian with an error of $\mathcal{O}(\Delta x^2)$. □

Since the non-linear reaction term $r(u)$ is polynomial, it is continuous and its derivative is also continuous. Furthermore, on any bounded interval, it is locally Lipschitz continuous. Due to these considerations, the approximating ODE (4) has a locally unique solution due to the Picard-Lindelöf theorem, see e.g. [22].

The $\mathcal{O}(\Delta x^2)$ consistency of the spatial discretization established above, together with [23, Theorem 3.1, Remark 3.1] proves that the solution of ODE (4) uniformly converges to the solution of the original Allen-Cahn PDE in each $\mathbf{x}_k$ point for $k = 0, 1, \dots, m^n$, as $\Delta x \to 0$.

III. Observer Design

Let the approximate ODE model given by the relation (4) that describes the evolution of the approximate states $\boldsymbol{\beta} \in \mathbb{R}^{\ell}_{+}$, where $\ell = m^n$.

If a number of p states of the system are measurable, $1 \le p \le \ell$, the measurement equation can be formulated as:

$$\boldsymbol{y} = C\boldsymbol{\beta} \quad (9)$$

where $\boldsymbol{y} \in \mathbb{R}^p_+$, is the output vector containing those entries of the state vector that can be measured. The output matrix $C \in \mathbb{R}^{p\times\ell}$ with entries 0 or 1 relates the output vector (measurements) to the corresponding states.

To estimate the unknown states of the approximate model, the following observer system is considered:

$$\dot{\widehat{\boldsymbol{\beta}}} = A\widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\beta}} \circ (\gamma\mathbf{1} - \widehat{\boldsymbol{\beta}}) \circ (\widehat{\boldsymbol{\beta}} - \theta\mathbf{1}) + \mathbf{f} + K(\mathbf{y} - C\widehat{\boldsymbol{\beta}}), \\ \widehat{\boldsymbol{\beta}}(0) = \widehat{\boldsymbol{\beta}}_0,\ \widehat{\boldsymbol{\beta}}_{0i} \in (0, \gamma) \quad (10)$$

where $\widehat{\boldsymbol{\beta}} \in \mathbb{R}^{\ell}$ denotes the estimated states, $K \in \mathbb{R}^{\ell\times p}$ is the observer gain matrix.

It is known that if the initial condition of the PDE satisfies $|u_0(\mathbf{x})| \le 1,\ \forall \mathbf{x} \in \Omega$ then the weak solution of the PDE is also bounded, i.e. $|u(\mathbf{x}, t)| \le \gamma$, see [17], Proposition 6.1. Since $f(\mathbf{x}, t) \in [0, \gamma)$, see Remark 1, β_M can be chosen as $\beta_M = \gamma$. In concordance with the PDE solution and due to bounded approximation errors, the states of the approximating system are also considered to be bounded, i.e. it is assumed that the inequality

$$0 < \beta_i < \beta_M \quad \forall i, \quad (11)$$

holds, where $\beta_M \in \mathbb{R}^*_+$ is an upper bound value.

Remark 3: As the ODE model state is always bounded, the observer can be extended using projection or set-bounding techniques so that the observer states remain bounded as well ($\mathbf{0} \le \widehat{\boldsymbol{\beta}} \le \boldsymbol{\beta}_M$). Specifically, by projecting the observer estimates onto a compact set consistent with the known bounds of the system, one can prevent excessive growth of the observer states due to uncertainties or high-gain injection, while preserving the convergence properties of the estimation error [24].

It is well-known from linear observers theory, see e.g. [11], that a gain matrix K can be designed such that the $A - KC$ is Hurwitz if the following observability condition holds:

$$\operatorname{rank}\begin{pmatrix} C \\ AC \\ \dots \\ A^{\ell-1}C \end{pmatrix} = \ell \quad (12)$$

As follows, it will be assumed that with the applied output matrix C the relation (12) is satisfied, and $A - KC$ is Hurwitz.

First, we analyze the performance of the estimator system by neglecting the uncertainty term in the model (4), i.e. $\mathbf{w} = \mathbf{0}$.

Define the estimation error:

$$\mathbf{e} = \boldsymbol{\beta} - \hat{\boldsymbol{\beta}}. \quad (13)$$

By equations (4) and (10), the dynamics of the observer error with $\mathbf{w} = \mathbf{0}$ is given by:

$$\dot{\mathbf{e}} = (A - KC)\mathbf{e} + \\ \Big((\gamma + \theta)(\boldsymbol{\beta}^2 - \hat{\boldsymbol{\beta}}^2) - \gamma\theta(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) - (\boldsymbol{\beta}^3 - \hat{\boldsymbol{\beta}}^3)\Big) \\ \dot{\mathbf{e}} = (A - KC)\mathbf{e} + \\ \Big((\gamma + \theta)(\boldsymbol{\beta} + \hat{\boldsymbol{\beta}}) - \gamma\theta\mathbf{1} - \boldsymbol{\beta}^2 - \boldsymbol{\beta} \circ \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^2\Big) \circ \mathbf{e} \quad (14)$$

Define the Lyapunov function candidate:

$$L = \frac{1}{2}\mathbf{e}^{\top}\mathbf{e} \quad (15)$$

Its time derivative is $\dot{L} = \mathbf{e}^{\top}\dot{\mathbf{e}}$. In the view of (14), it reads as:

$$\dot{L} = \mathbf{e}^T(A - KC)\mathbf{e} - \mathbf{e}^T\operatorname{diag}(\boldsymbol{\beta}^2)\mathbf{e} \\ + \mathbf{e}^T\operatorname{diag}\left(\underbrace{(\gamma + \theta)(\boldsymbol{\beta} + \hat{\boldsymbol{\beta}}) - \gamma\theta\mathbf{1} - \boldsymbol{\beta} \circ \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^2}_{\mathbf{b}(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}})}\right)\mathbf{e} \quad (16)$$

It is explored that

$$\mathbf{e}^T(A-KC)\mathbf{e} \leq \lambda_M \mathbf{e}^T\mathbf{e},$$

where λ_M is the rightmost eigenvalue of $\mathrm{sym}(A-KC)$. It is considered that K is designed so that λ_M is always strictly negative, e.g. based on the LMI (Linear Matrix Inequality) $\mathrm{sym}(A-KC) < \alpha I$ for some desired $\alpha > 0$. It follows that

$$\dot{L} < \lambda_M \mathbf{e}^T\mathbf{e} + \max(\mathbf{b})\mathbf{e}^T\mathbf{e} \tag{17}$$

where $\max(\mathbf{b})$ is the rightmost entry of $\mathbf{b}(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}})$ defined in (16).

To obtain an upper bound for $\mathbf{b}$, we leverage the fact that the system states are bounded ($\mathbf{0} \leq \boldsymbol{\beta} \leq \boldsymbol{\beta}_M$). Then the following bound yields:

$$\mathbf{b}_M = (\gamma+\theta)(\beta_M\mathbf{1} + \hat{\boldsymbol{\beta}}) - \gamma\theta\mathbf{1} - \hat{\boldsymbol{\beta}}^2 \geq \mathbf{b} \tag{18}$$

For the cases when the boundedness of the observer estimates is also ensured through the use of appropriate state projection techniques [24], i.e. $\mathbf{0} \leq \hat{\boldsymbol{\beta}} \leq \boldsymbol{\beta}_M$, we obtain the following constant bound:

$$\bar{b}_M = 2(\gamma+\theta)\boldsymbol{\beta}_M - \gamma\theta \geq \max(\mathbf{b}_M) \tag{19}$$

In the view of (17), when $\max(\mathbf{b}_M) \leq 0$, the Lyapunov function decreases $\forall\ \lambda_M < 0$.

Otherwise, the observer gain K has to be designed such that $|\lambda_M| > \max(\mathbf{b}_M)$.

Consider a switching gain in the form:

$$K = \begin{cases} K_n, & \text{if } \max(\mathbf{b}_M) < 0 \\ K_s, & \text{otherwise.} \end{cases} \tag{20}$$

One can choose the nominal gain K_n primarily for performance, e.g. noise rejection, desired convergence rate, using standard linear observer design techniques.

Otherwise, the observer has to switch to a high gain (K_s) to deal with additional energy injection [25]. It can be computed by considering that in the critical regime we have:

$$\dot{L} \leq \mathbf{e}^T\mathrm{sym}(A-K_sC)\mathbf{e} + \max(\mathbf{b}_M)\mathbf{e}^T\mathbf{e} \tag{21}$$

Hence, the following LMI can be applied to compute K_s:

$$\mathrm{sym}(A-K_sC) < \bar{b}_M I \tag{22}$$

The resulting observer structure ensures the negativeness of $\dot{L}$ for all $\mathbf{e} \neq \mathbf{0}$. By Lyapunov's theorem, it yields that $\lim_{t\to\infty}\|\mathbf{e}(t)\| = 0$, where $\|\cdot\|$ is a norm on $\mathbb{R}^{\ell}$.

Summary of the observer design: Let the ODE model (4) with the output equation (9) such that (12) is satisfied. Then the observer system (10) in which the observer gain defined by (20) ensures the convergence of the estimation error to zero.

Remark 4: The approximation error term $\mathbf{w}$ can be viewed as a process noise in the model (4) applied to observer design. Moreover, it can also be considered that the measurement equation (9) is also affected by sensor noise($\mathbf{v}(t)$), i.e. $\mathbf{y} = C\boldsymbol{\beta} + \mathbf{v}(t)$. To take into account these uncertainties, the nominal observer gain K_n can be designed by using e.g. the such estimator design which ensures noise attenuation [11].

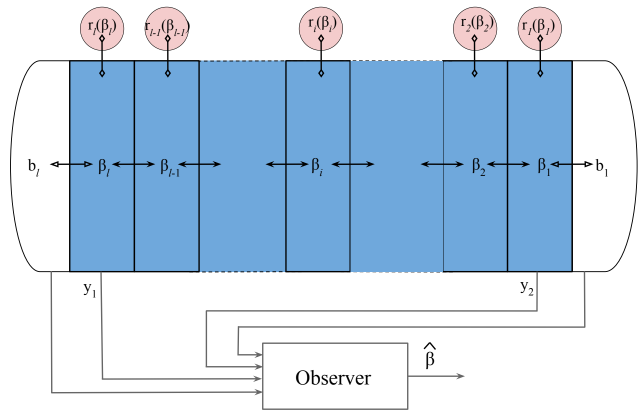

Fig. 1. The spatially discretized model with observer ($n = 1$, $p = 2$)

The compartmental view of the approximation model for the one-dimensional case is presented in Fig. 1. As it can be seen, the boundary conditions directly influence only the dynamics of the first and the lth states.

IV. Case study

The usability of the proposed approximation-based PDE solution estimator was tested through simulations. The PDE model and the proposed observer were implemented in a Python environment.

We set the spatial domain $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$, where $L = 1$, with an interior cross-like boundary, as shown in red in Fig. 2. We have imposed a homogeneous Dirichlet boundary condition, i.e., the solution is zero on the spatial boundary $\partial\Omega$. Let $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top$ be the spatial coordinates. The simulation time range is $t = [0, 20]$ seconds. To get the solution of the PDE given by Eq. (1), we applied a Forward-Time Central-Space (FTCS) method.

We store each state value $u_{i,j}$ in a 1-dimensional vector with row major ordering: $r \to i \cdot N_x + j$ where $N_x = L/\Delta x$ is the number of cells per dimension. The lattice resolution was chosen as $\Delta x = 100$. We define the Laplacian-matrix as:

$$A = q(K \otimes I + I \otimes T) \in \mathbb{R}^{N_x^2 \times N_x^2} \tag{23}$$

The matrices are defined as $T = \mathrm{TriDiag}(1, -4, 1) \in \mathbb{R}^{N_x \times N_x}$ and $K = \mathrm{TriDiag}(1, 0, 1) \in \mathbb{R}^{N_x \times N_x}$. The notation $\mathrm{TriDiag}(B_{-1}, B_0, B_{+1})$ denotes tridiagonal matrix with sub-, main, and super-diagonal entries respectively.

We applied a slowly moving Gaussian source term:

$$f(t, \boldsymbol{x}) = e^{-200\cdot\left[(x_1 - C_1(t))^2 + (x_2 - C_2(t))^2\right]} \tag{24}$$

where $C_1 = L/2 + r\cos(\omega t)$, $C_2 = L/2 + r\sin(\omega t)$ and $r = 0.3$, $\omega = 0.05\pi$. This results in a moving Gaussian source term shown in blue in Figures 2 and 3 which is vanishing in each point of its trajectory.

Normally distributed measurement and process noises were considered in the model: $w_k, v_k \sim \mathcal{N}(0, 0.01)$. The measurement is defined as $y_k^m = u_k \cdot s_k + w_k$.

$$y_k^m = u_k \cdot s_k + w_k \tag{25}$$

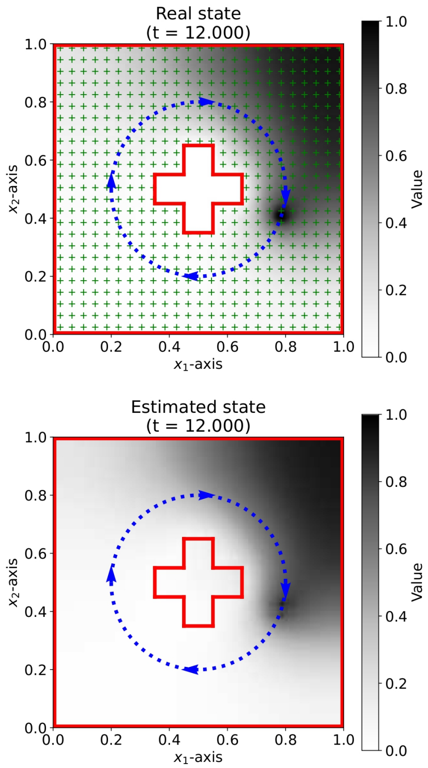

Fig. 2. Ideal model and observer at time $t = 12$ seconds ($\theta = 0.1$)

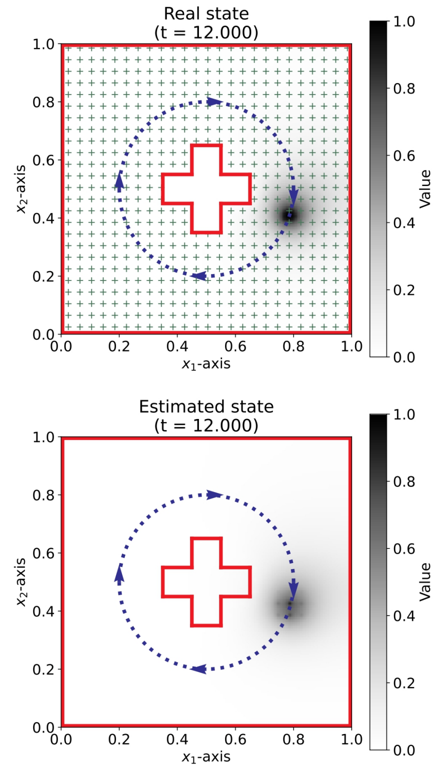

Fig. 3. Ideal model and observer at time $t = 12$ seconds ($\theta = 0.9$)

Here $s_k \in \{0, 1\}$ is a binary sensor mask: $s_k = 1$ if the k-th lattice node is a measurement location, and $s_k = 0$ otherwise, consistent with the non-zero rows of the output matrix C.

We consider a sensor mask, that forms a lattice with coarser spatial resolution than that of the original discretization, shown in green in Fig. 2, 3. Specifically, if the inter-sensor distance is s_d, a sensor is placed at every s_d-th lattice point along each spatial dimension of the discretized domain. The output matrix C is defined according to this sensor mask, and its dimension is $N_x/s_d \times N_x/s_d$.

The observability for the (A, C) matrix pair was checked using the `obsv` function. The observability condition (12) with the considered output matrix stands.

We present two simulation scenarios. In accordance with *Remark 2*, we consider two cases corresponding to small and large values of the parameter θ. In the first case, when θ is small, the solution converges to the value γ. In contrast, for large values of θ, the solution converges to zero.

In the first simulation scenario, the parameters of the reaction term are set to $\theta = 0.1$ and $\gamma = 1$, with an inter-sensor distance of $s_d = 4$. As shown in Fig. 2, the PDE solution exhibits a monotonic increasing behavior and asymptotically saturates at the value γ. The figure also demonstrates that the estimated state closely tracks the true system state.

In the second simulation scenario, the reaction term parameters are chosen as $\theta = 0.9$ and $\gamma = 1$, while maintaining the inter-sensor distance at $s_d = 4$. In this case, the PDE solution

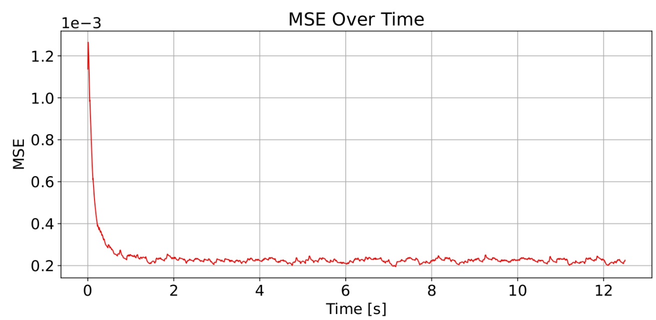

Fig. 4. Observer error ($\theta = 0.9$)

decreases in each spatial compartment, as illustrated in Fig. 3.

The convergence of the estimation error is illustrated in Fig. 4, which shows the mean square error (MSE) of the vector $\mathbf{e}$ defined in (13) across the entire discretized state space, highlighting the rapid and accurate convergence of the observer.

We also evaluated the influence of inter-sensor distance on the overall observer error. Let $s_d \in [2, 10]\,\Delta x$.

Fig. 5 shows the MSE of the estimation error as a function of s_d. As expected, the MSE increases with larger inter-sensor distances.

Despite the markedly different dynamic behaviors of the PDE solution under different parameterizations, the simulation

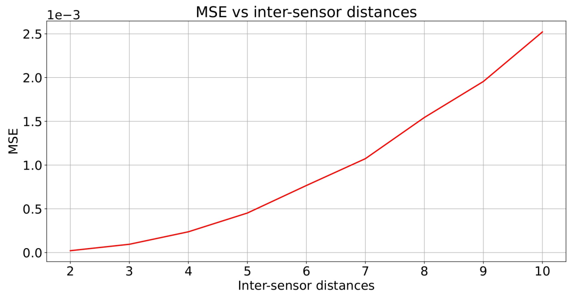

Fig. 5. Inter-sensor distance versus observer MSE ($\theta = 0.9$).

results demonstrate that the observer performs effectively in all cases, relying solely on the defined measurement points.

V. Conclusions

A novel observer design method has been developed for a class of reaction–diffusion systems, based on a time-continuous approximate model obtained via the method of lines. To address the challenges introduced by the sign-changing nonlinear reaction term, a switching observer gain is incorporated into the observer structure. Under nominal operating conditions, the gain can be designed to ensure effective noise attenuation, whereas in critical regimes a high-gain strategy is employed to guarantee convergence of the estimation error to zero over the entire spatial domain. The proposed observer is computationally efficient, as it avoids direct implementation of the original partial differential equation. Its dimensionality depends solely on the chosen spatial discretization resolution. The observability condition is derived from the classical rank condition of the observability matrix, commonly used in linear dynamical systems. Simulation results validate the effectiveness of the proposed observer, demonstrating accurate estimation of the system state. Overall, this work provides a meaningful contribution to state estimation problems in reaction–diffusion systems, with potential applications in areas such as theoretical ecology. As a direction for future research, the observer framework may be extended by incorporating an unknown input disturbance estimation term to compensate for the nonzero estimation offset introduced by numerical approximation errors.

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